From Primitive Global Non-Distinction to Relational Causality: A Conditional Construction of 1 + 1-Dimensional Lorentzian Pregeometry

Chaim Zeitz
Affiliation: HDIF Nexus Research Labs
Website: https://chaimz2020.wixsite.com/hdif-nexus
Independent Researcher, Tamarac, United States
Corresponding author: ZeitzChaim@gmail.com
ORCID: 0009-0000-7129-0349
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Abstract

Contemporary physical theories presuppose distinguishable states, relations, or geometric structure. This paper asks whether an effectively differentiated causal geometry can be constructed from a primitive description containing no intrinsic differentiation. Primitive Global Non-Distinction (PGND) is represented by a single ground that, under an explicit articulation principle, occupies two non-equivalent relational positions without becoming two primitive occupants. Conditional on independently stated principles of persistent consequence, recentering, two-orientation articulation, and repeatable-extension closure, finite compositions form a binary history monoid. Count-based endpoint recombination yields an endpoint space isomorphic to the nonnegative integer lattice, and extension-induced reachability defines a locally finite product order. After causal and metric specialization, this order is isomorphic to the causal order induced on a regular null lattice in $1 + 1$-dimensional Minkowski spacetime. At fixed lattice scale the construction provides timelike, null, and spacelike classification, causal diamonds, a proper-time function, and quadratic interval growth. Under refinement, rescaled interval counting converges to the corresponding Alexandrov-volume law; independently, the large-interval ordering fraction approaches the $1 + 1$-dimensional Myrheim–Meyer value. Countermodels isolate the independent work performed by the bridge principles and physical specializations. A finite-generator obstruction further shows that a fixed finite alphabet of independent endpoint directions cannot exactly generate the non-polyhedral Lorentz cone in spacetime dimension three or greater. The result is therefore a conditional admissibility witness, not a derivation

from PGND alone of physical realization, uniqueness, higher-dimensional spacetime, microscopic Lorentz symmetry, dynamics, quantum structure, or empirical predictions.

Keywords: Primitive Global Non-Distinction, relational pregeometry, causal order, discrete spacetime, Lorentzian geometry, emergent spacetime

1 Introduction

Contemporary physical theories presuppose mathematically distinguishable structure. General relativity begins with a differentiable manifold equipped with geometric and causal relations, while quantum theory begins with state spaces, observables, amplitudes, and distinguishable alternatives. These frameworks provide extraordinarily successful descriptions of physical phenomena, but they do not by themselves answer a logically prior question: what is the weakest structural starting point from which relationally effective distinctions can be represented without assuming distinguishable objects, background geometry, or temporal ordering at the outset?

This question motivates the hypothesis of *Primitive Global Non-Distinction* (PGND). At the primitive descriptive level, PGND represents no intrinsic differentiation among objects, positions, moments, observables, or physical degrees of freedom. It should not be identified in advance with zero, infinity, an empty set, a quantum state, or a maximally symmetric spacetime, since each of these already belongs to a mathematical framework containing distinctions and operations richer than the primitive hypothesis. The primitive condition is therefore denoted abstractly by

$$G, \tag{1}$$

without initially assigning to G a set-theoretic, algebraic, geometric, probabilistic, or dynamical interpretation.

The problem is not merely to attach a familiar mathematical object to G. Any proposed representation already introduces rules of identity, relation, composition, and transformation. The foundational task is consequently to identify which additional structures are required for effective relational distinction and to prevent those structures from being attributed retrospectively to the primitive level. In particular, the existence of a relationally differentiated representation does not imply that its distinctions were intrinsic constituents of G.

The first additional structure considered here is self-presentation. Schematically,

$$R_{\mathrm{sp}} : G \longmapsto \big(G_{\mathrm{pr}}, G_{\mathrm{pd}}\big), \tag{2}$$

where pr and pd denote presenting and presented relational positions. The two positions are occupied by the same primitive ground:

$$\omega\big(G_{\mathrm{pr}}\big) = \omega\big(G_{\mathrm{pd}}\big) = G, \tag{3}$$

while their relational roles are formally non-equivalent:

$$\mathrm{pr} \neq \mathrm{pd}. \tag{4}$$

Equation (2) therefore does not introduce two intrinsically distinct primitive objects. It introduces two relational positions occupied by one primitive occupant, separating identity of primitive ground from non-equivalence of relational position.

Self-presentation is not treated as a theorem of existence alone. If "presentation" meant only that G exists as itself, the claim would be analytic but structurally unproductive. If it introduces presenting and presented positions, it contributes relational structure not contained in bare identity $G = G$. Minimal self-presentation is therefore stated explicitly as an articulation principle of the construction rather than as a consequence of PGND alone.

Nor does self-presentation by itself generate an extended history space. The later construction requires persistent consequence, recentering, a two-orientation articulation, and the Repeatable-Extension Closure Principle. The two relational orientations furnish the alphabet

$$B = \{0, 1\}. \tag{5}$$

Repeatable-extension closure then admits the empty history and both orientation extensions after every admitted finite history. The resulting kinematic history space is

$$\mathcal{H} = B^* = \{0, 1\}^*. \tag{6}$$

The symbols 0 and 1 encode the two non-absolute orientations of the presenting–presented articulation rather than two primitive substances. A word

$$w = b_1 b_2 \cdots b_n, \qquad b_i \in B, \tag{7}$$

records a finite relational composition. It does not yet represent a trajectory through antecedently existing spacetime.

The central question of the paper is therefore:

> *Can a same-occupant relational articulation, together with explicitly stated principles of persistence, recentering, two-orientation articulation, repeatable-extension closure, endpoint recombination, reachability, and physical specialization, support a locally finite causal structure with a calculable* $1 + 1$*-dimensional Lorentzian interpretation, without treating the resulting distinctions as intrinsic divisions of the primitive ground?*

Throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$.

To answer this question, define the endpoint-count map

$$q : \mathcal{H} \longrightarrow \mathbb{N}^2, \qquad q(w) = \big(N_0(w), N_1(w)\big), \tag{8}$$

where $N_i(w)$ counts the occurrences of the orientation i in w. The Endpoint-Recombination Principle identifies histories having the same endpoint counts:

$$w \sim_q w' \quad \Longleftrightarrow \quad q(w) = q(w'). \tag{9}$$

The endpoint space is consequently the quotient

$$\mathcal{E} = \mathcal{H}/\!\sim_q. \tag{10}$$

Let

$$\pi_q : \mathcal{H} \longrightarrow \mathcal{E}, \qquad \pi_q(w) = [w]_q, \tag{11}$$

denote the quotient map. The count map induces a monoid isomorphism

$$\overline{q} : \mathcal{E} \xrightarrow{\sim} \mathbb{N}^2, \qquad \overline{q}([w]_q) = q(w). \tag{12}$$

The quotient records effective endpoint convergence without asserting total identity of the underlying histories. Thus

$$q(w) = q(w') \tag{13}$$

need not imply

$$w = w'. \tag{14}$$

Histories may converge to the same endpoint while remaining distinct in ordered provenance or surviving internal constraint.

Further endpoint extension induces the coordinatewise reachability relation

$$(u, v) \preceq_R (u', v') \quad \Longleftrightarrow \quad u \leq u' \text{ and } v \leq v'. \tag{15}$$

The paper proves that this relation is reflexive, antisymmetric, transitive, and locally finite. Local finiteness is therefore an order-theoretic consequence of the endpoint quotient and extension-induced reachability. The additional locality condition performs a different role: it identifies cover relations, rather than arbitrarily large coordinate jumps, as elementary propagation steps.

The ordered endpoint structure is not intrinsically Lorentzian. A Lorentzian interpretation is introduced only through a further physical specialization. For a uniform positive scale

$$\ell > 0, \tag{16}$$

define

$$\iota_\ell : \mathcal{E} \longrightarrow \mathbb{R}^{1,1} \tag{17}$$

by

$$\iota_\ell(u, v) = (T, X), \qquad T = \ell(u + v), \qquad X = \ell(v - u). \tag{18}$$

Its image is the regular discrete future wedge

$$\mathbb{M}^{1,1}_{\ell,+} = \iota_\ell(\mathcal{E}). \tag{19}$$

Under the Minkowski metric,

$$ds^2 = -dT^2 + dX^2 = -4\, dU\, dV, \qquad U = \ell u, \quad V = \ell v, \tag{20}$$

the coordinates u and v, after multiplication by ℓ, function as discrete null coordinates. This null interpretation is not projected backward onto the primitive labels 0 and 1; it arises only under the Minkowski specialization.

The embedding is an order isomorphism between $(\mathcal{E}, \preceq_R)$ and the causal order induced on $\mathbb{M}^{1,1}_{\ell,+}$. With a uniform endpoint weight $2\ell^2$, rescaled interval counting converges to the corresponding Minkowski Alexandrov-volume law in the refinement limit. The construction therefore provides an explicit $1+1$-dimensional Lorentzian causal pregeometry and a calculable admissibility witness for the compatibility of primitive global non-distinction with effective relational distinction.

Minimal PGND is not a generative axiom for the history monoid, endpoint quotient, causal order, or Minkowski specialization. Its role is restrictive: it fixes the primitive-description constraint under which the construction must operate by excluding intrinsic differentiation from the primitive ground. The bridge principles supply the positive relational structure. Accordingly, the result establishes that a discrete Lorentzian causal pregeometry is compatible with PGND under sufficient additional conditions; it does not establish that this structure follows from PGND alone.

The claimed contribution does not lie in the product order on $\mathbb{N}^2$, its realization as a regular $1+1$-dimensional null lattice, or the fact that a finitely generated cone cannot equal a non-polyhedral higher-dimensional Lorentz cone. These are established mathematical structures and results used here as diagnostic targets. The contribution is the explicit same-occupant relational construction leading to them, together with the dependency and countermodel audit that identifies the additional principle required at each stage. The construction therefore provides one PGND-compatible admissibility witness while leaving open whether different principles of articulation, extension, recombination, locality, orientation, or metric scale could produce other PGND-compatible pregeometries.

The construction can be situated more precisely in relation to existing approaches. Abstract causal spaces provide an order-theoretic language in which causal precedence can be studied without taking metric distance as primitive [1]. Most directly, the present construction intersects with causal set theory, which represents discrete spacetime by a locally finite partially ordered set whose order relation carries proto-causal significance. The causal-set program ordinarily begins with causal order and local finiteness as fundamental structural inputs, often summarized as the combination of order and number [2, 3].

The methodological relation is complementary: whereas causal set theory begins with a locally finite causal order, the present construction investigates one route by which such an order can arise from a same-occupant articulation, a relational history space, and an endpoint quotient. The stages of this construction are schematically

$$G \xrightarrow{R_{\mathrm{sp}}} \big(G_{\mathrm{pr}}, G_{\mathrm{pd}}\big) \xrightarrow{\mathrm{Ext}^*} \mathcal{H} \xrightarrow{\pi_q} \mathcal{E} \xrightarrow{\preceq_R} (\mathcal{E}, \preceq_R) \qquad \mathcal{H} \xrightarrow{\Pi_G} \{G\} \tag{21}$$

followed by the physical specialization

$$(\mathcal{E}, \preceq_R) \xrightarrow{\iota_\ell} \mathbb{M}^{1,1}_{\ell,+}. \tag{22}$$

Here Ext^* schematically abbreviates the combined use of persistent consequence, recentering, the two-orientation articulation, and repeatable-extension closure. It is not a consequence of R_{sp} alone. Likewise, π_q, $\preceq_R$, and ι_ℓ represent logically distinct stages rather than one undifferentiated transition from G to spacetime.

The relevance of causal order to Lorentzian geometry is supported by reconstruction results showing that, subject to appropriate causality conditions, causal structure determines the conformal geometry of a spacetime, while additional volume information is required to fix the remaining conformal factor [4, 5]. These results motivate the separation maintained throughout this paper:

$$\text{causal order} \longrightarrow \text{conformal Lorentzian structure} \longrightarrow \text{metric scale}. \tag{23}$$

The arrows in Eq. (23) are conditional and should not be conflated. A discrete causal order is not by itself a continuum spacetime, and correct interval growth does not by itself establish exact microscopic Lorentz invariance.

The framework is also related in spirit to relational and information-oriented approaches to physical foundations. Relational quantum mechanics, for example, treats physical facts as relational rather than as universally possessed intrinsic properties [6]. Information-first programs similarly investigate whether physical structure can be understood through distinctions, correlations, or acts of registration rather than through independently given substances [7]. More broadly, research on pregeometry and emergent spacetime examines what must be recovered when continuum spatial and temporal structure is absent from the fundamental description [8, 9].

The present construction belongs to this broad problem domain but does not adopt the specific microscopic ontology, quantum formalism, or emergence mechanism of these approaches. PGND does not begin with distinguishable systems, quantum states, observers, measurements, or classical bits. The binary labels introduced here encode relational orientations within one same-occupant articulation; they do not initially denote two intrinsically different primitive alternatives.

The construction is also kinematical rather than dynamical. None of the maps above is interpreted as temporal evolution, symmetry breaking, or cosmological production. They specify relations among successively enriched descriptive structures. A physical generative interpretation would require additional principles selecting and dynamically realizing particular histories. No such selection law is derived here.

Against this background, and subject to the explicitly stated bridge principles and physical specializations, the paper makes four contributions to the analysis of PGND-compatible relational construction.

1. It formulates self-presentation as a same-occupant, two-position relational articulation and distinguishes reversible role exchange from persistent extension. Under the additional persistence, recentering, two-orientation,

and repeatable-extension principles, this yields the binary history monoid

$$\mathcal{H} = \{0, 1\}^*.$$

2. It makes explicit the passage from ordered relational histories to endpoint structure. Count-based endpoint equivalence is shown to be a monoid congruence, giving

   $$\mathcal{H}/\sim_q \cong (\mathbb{N}^2, +),$$

   after which extension-induced reachability yields the locally finite product order on the endpoint space. The contribution is the audited derivation of this familiar structure from the stated relational premises, not the product order itself.

3. It uses the standard null-coordinate realization of the product order to identify the resulting endpoint poset with a regular discrete 1+1-dimensional Minkowski future wedge. Within this specialization, it calculates the causal classification, interval-cardinality law, continuum counting convergence, and asymptotic agreement with the $1 + 1$-dimensional Myrheim–Meyer ordering fraction. These calculations characterize the constructed model rather than assert a new null-lattice geometry.

4. It supplies countermodels that isolate the work performed by the bridge principles and prevent the resulting structures from being attributed to PGND alone. It then applies the standard distinction between finitely generated polyhedral cones and non-polyhedral Lorentz cones to show why this construction, when restricted to a fixed finite collection of independent endpoint directions, does not extend exactly to spacetime dimension $D \geq 3$.

Collectively, these contributions establish a conditional existence and admissibility result for one PGND-compatible discrete Lorentzian causal pregeometry. The claimed contribution is the explicit construction and its dependency audit, not the independent discovery of the product-order null lattice or the finite-generator obstruction. The results do not establish unique determination, physical realization, dynamical selection, exact microscopic Lorentz invariance, or extension to the complete structure of the observed $3 + 1$-dimensional universe.

Methodologically, to prevent stronger conclusions from being hidden in definitions, every substantive claim is assigned one of five logical statuses:

1. *definition*: notation or stipulated mathematical meaning;

2. *derived result*: a consequence proved from previously stated premises;

3. *bridge principle*: additional structural content required for the construction to advance;

4. *physical specialization*: a physical interpretation assigned to an abstract relational structure;

5. *open problem*: a proposed connection not established by the present results.

Section 8 uses countermodels to distinguish primitive compatibility from unique articulation, self-presentation from persistent extension, binary history from endpoint recombination, endpoint convergence from complete history identity, and Lorentzian causal form from exact microscopic Lorentz invariance. The resulting construction is conditional but nontrivial: once its premises are stated, their mathematical consequences are explicit and calculable, while the countermodels show why those premises cannot be attributed silently to PGND alone.

2 Primitive Non-Distinction and Minimal Articulation

This section distinguishes Minimal from Strong PGND and tests whether primitive non-distinction is compatible with relational articulation at a non-primitive level. Minimal Self-Presentation is introduced as an explicit bridge from one primitive occupant to two non-equivalent relational positions, whose non-absoluteness is represented by mirror exchange.

2.1 Primitive Global Non-Distinction

Let

$$G \tag{24}$$

denote the primitive globally non-distinguished condition. The symbol G is initially only a placeholder for the primitive descriptive level. It does not yet denote a set, algebra, state vector, manifold, field, probability distribution, or physical state evolving in time.

Two interpretations of PGND must be distinguished.

Definition 2.1 (Minimal PGND)**.** *Minimal Primitive Global Non-Distinction, denoted $G_{\min}$, states that no intrinsic differentiation is represented at the primitive descriptive level:*

$$G_{\min}: \qquad \textit{no primitive intrinsic distinction is represented.} \tag{25}$$

Minimal PGND is a claim about the structural content of the primitive description. It does not assert that no richer relational description can exist.

Definition 2.2 (Strong PGND)**.** *Strong Primitive Global Non-Distinction, denoted G_{strong}, states that no intrinsic differentiation exists at the primitive ontological level:*

$$G_{\text{strong}}: \qquad \textit{no primitive intrinsic distinction exists.} \tag{26}$$

Strong PGND entails a more substantial ontological commitment than Minimal PGND. In particular,

$$G_{\text{min}} \not\Rightarrow G_{\text{strong}}. \tag{27}$$

Unless a result explicitly requires the stronger interpretation, the present construction proceeds from Minimal PGND.

At the primitive level, the framework therefore assumes neither

$$\{G_1, G_2, \ldots\} \tag{28}$$

nor any prior structure capable of distinguishing primitive components. In particular, the primitive description does not presuppose space, time, distance, direction, metric, causal order, observables, probability, interaction, or physical dynamics.

These exclusions are methodological rather than empirical. They specify the descriptive content removed in the search for an assumption-minimal starting point.

2.2 Cross-Level Non-Exclusion

Let N_G denote the absence of represented intrinsic differentiation at the primitive descriptive level, and let D_R denote effective distinction among positions, histories, or endpoints at a relationally articulated level. These conditions concern different structural levels and are therefore not defined here as logical complements:

$$D_R \neq \neg N_G. \tag{29}$$

It does not follow merely from this level distinction, however, that N_G and D_R are jointly realizable.

The relevant compatibility question is consequently constructive:

> Can there exist a relational structure that exhibits effective relational distinction while all of its differentiated positions remain grounded in one primitive occupant for which no intrinsic differentiation is represented?

Cross-Level Non-Exclusion is therefore not assumed at this stage as an established consistency theorem. It is the compatibility claim to be tested by the explicit construction developed below. The construction must provide a witness satisfying both of the following conditions:

1. no plurality of intrinsically differentiated primitive occupants is introduced; and

2. non-equivalent relational positions or consequences are nevertheless represented.

Such a witness would establish the conditional joint representability of N_G and D_R, not

$$N_G \Longrightarrow D_R. \tag{30}$$

Cross-Level Non-Exclusion therefore concerns compatibility under the stated bridge principles, not derivation from PGND or physical realization.

As a limited logical illustration, classical double negation gives

$$N_G \equiv \neg\neg N_G.$$

The expressions N_G and $\neg\neg N_G$ are syntactically non-identical while having the same classical truth conditions. This illustrates that descriptive non-identity need not entail a plurality of primitive referents. It does not generate presenting and presented positions, provide evidence for self-presentation, or supply any dynamical or structural bridge used in the construction.

2.3 The Self-Presentation Bridge

Bare identity gives

$$G = G. \tag{31}$$

Equation (31) expresses self-identity but contains no differentiated relational positions. It therefore does not by itself establish self-presentation in the structurally productive sense required by the construction.

The first additional principle is stated explicitly.

Minimal Self-Presentation Principle. The primitive condition G admits an articulation in which the same primitive occupant occurs in presenting and presented relational positions:

$$R_{\mathrm{sp}}(G) := \big(G_{\mathrm{pr}}, G_{\mathrm{pd}}\big). \tag{32}$$

The presenting and presented positions are relationally non-equivalent:

$$G_{\mathrm{pr}} \neq_{\mathrm{role}} G_{\mathrm{pd}}, \tag{33}$$

but they are occupied by the same primitive ground. If Occ denotes the set of primitive occupants represented in an articulation, then

$$\mathrm{Occ}\big(R_{\mathrm{sp}}(G)\big) = \{G\}, \tag{34}$$

and hence

$$\#\,\mathrm{Occ}\big(R_{\mathrm{sp}}(G)\big) = 1. \tag{35}$$

Equation (32) is an articulation bridge, not a consequence of the identity $G = G$. It introduces two non-equivalent relational positions while preserving the singleton occupant set established by Eq. (35). Its minimality therefore concerns primitive-occupant cardinality; it establishes neither that self-presentation is uniquely selected nor that its roles generate persistent extension.

2.4 One Occupant and Two Relational Positions

Define the articulated position set

$$A_0 = \{G_{\mathrm{pr}}, G_{\mathrm{pd}}\}\,. \tag{36}$$

Define the primitive-occupant map

$$\omega : A_0 \longrightarrow \{G\} \tag{37}$$

by

$$\omega\big(G_{\mathrm{pr}}\big) = \omega\big(G_{\mathrm{pd}}\big) = G. \tag{38}$$

The positions are non-equivalent as relational roles:

$$G_{\mathrm{pr}} \neq_{\mathrm{role}} G_{\mathrm{pd}}, \tag{39}$$

while they remain identical with respect to primitive occupant:

$$G_{\mathrm{pr}} \equiv_{G} G_{\mathrm{pd}}. \tag{40}$$

Equations (39) and (40) express the central separation:

$$\boxed{\text{one primitive occupant} + \text{two relational positions.}} \tag{41}$$

Proposition 2.1 (Relational position does not entail primitive plurality)**.** *The existence of two non-equivalent relational positions does not entail the existence of two intrinsically distinct primitive occupants.*

Proof. Consider the articulated position set A_0 and occupant map ω defined in Eqs. (36)–(38). The positions satisfy

$$G_{\mathrm{pr}} \neq_{\mathrm{role}} G_{\mathrm{pd}}, \tag{42}$$

while

$$\omega\big(G_{\mathrm{pr}}\big) = \omega\big(G_{\mathrm{pd}}\big) = G. \tag{43}$$

Hence positional multiplicity is compatible with a singleton primitive-occupant set. Therefore,

$$\text{relational-position plurality} \not\Rightarrow \text{primitive-object plurality}. \tag{44}$$

□

Proposition 2.1 establishes compatibility between relational-position plurality and a singleton primitive-occupant set, not the necessity of that articulation.

2.5 Non-Absolute Position and Mirror Exchange

Presenting and presented are relational positions rather than intrinsic identities. Define the role-exchange operation

$$J : A_0 \longrightarrow A_0 \tag{45}$$

by

$$J\big(G_{\mathrm{pr}}\big) = G_{\mathrm{pd}}, \qquad J\big(G_{\mathrm{pd}}\big) = G_{\mathrm{pr}}. \tag{46}$$

Applying the exchange twice restores the original role assignment:

$$J^2 = I_{A_0}, \tag{47}$$

where I_{A_0} is the identity on A_0.

The exchange preserves primitive occupancy:

$$\omega(Jx) = \omega(x) \qquad \text{for every } x \in A_0. \tag{48}$$

Thus J changes relational orientation without changing primitive ground. The pair

$$\big(A_0, J\big) \tag{49}$$

carries a minimal $\mathbb{Z}_2$-exchange symmetry.

This symmetry does not yet constitute spatial reflection, time reversal, causal reversal, or a Lorentz transformation. Those interpretations require geometric and physical structures not contained in A_0.

Mirror exchange must also be distinguished from persistent extension. The involution satisfies

$$J^2 = I, \tag{50}$$

whereas an extension operation E, if later established, must permit

$$E^n \neq I \qquad \text{for at least some } n > 0. \tag{51}$$

The distinction between J and E will become essential when the relational history space is constructed.

2.6 Effective Relational Distinction

Role labels alone need not produce an operationally meaningful distinction. Let

$$C(x) \tag{52}$$

denote the relational consequences associated with position $x \in A_0$.

Define effective relational distinction by

$$D_R(x, y) \quad \Longleftrightarrow \quad C(x) \neq C(y). \tag{53}$$

Accordingly,

$$\omega(x) = \omega(y) = G, \qquad C(x) \neq C(y)$$

is sufficient for $D_R(x, y)$ without primitive-occupant plurality. This is a compatibility condition, not a construction of the consequence assignment C. The later extension maps

$$E_0(w) = w0, \qquad E_1(w) = w1$$

provide a concrete realization of distinct relational consequences under the additional persistence, orientation, and extension principles.

2.7 Status Ledger

Table 1 summarizes the logical status of the structures introduced in this section.

Table 1: Logical status of the primitive articulation

Structure or claim	**Logical status**
Primitive Global Non-Distinction G	Primitive hypothesis
Minimal and Strong PGND	Distinct interpretations
Cross-Level Non-Exclusion	Compatibility condition
Minimal self-presentation	Explicit articulation bridge
Presenting and presented positions	Defined within the articulation
Common primitive occupant	Part of the articulation hypothesis
Role exchange J	Defined involution
$J^2 = I$	Derived from the definition of J
Relational-position plurality without primitive plurality	Consistency proposition
Effective relational distinction	Defined by non-equivalent relational consequences; its occurrence requires additional bridge principles
Physical causal or geometric interpretation	Not yet introduced

The next section introduces persistent consequence, recentering, and repeatable-extension closure and uses these principles to construct the kinematically admitted binary history space. It will remain essential to distinguish the reversible role exchange J from the nontrivial extension operations required to generate relational histories.

3 Persistent Extension and the Relational History Space

This section introduces the conditions under which a same-occupant articulation can leave persistent consequences and undergo repeated extension. These conditions generate the binary history monoid while preserving the histories' provenance and mirror structure. Endpoint identification and causal interpretation are introduced only in later sections.

3.1 Occurrence and Persistent Consequence

The preceding section established only a same-occupant, two-position articulation:

$$R_{\mathrm{sp}}(G) = \big(G_{\mathrm{pr}}, G_{\mathrm{pd}}\big). \tag{54}$$

A single articulation does not yet generate an extended relational structure. For extension to be represented, some consequence of an established articulation must remain structurally available for subsequent composition.

It is necessary to distinguish the proposition *the articulation occurred* from the stronger proposition *a consequence of the articulation remains structurally available*.

Let e denote one completed presenting–presented articulation. A persistent trace is represented by

$$\tau(e), \tag{55}$$

where $\tau(e)$ denotes whatever structure allows the completed articulation to remain consequential for a later admissible composition.

The required bridge is stated explicitly.

Persistent Consequence Principle. Every completed articulation admitted within the kinematic construction leaves a nonempty structural consequence:

$$\tau(e) \neq \varnothing. \tag{56}$$

This consequence remains structurally available for at least one further admissible relational composition.

The word "persistent" does not initially denote persistence through physical time. It means only that the consequence of e remains part of the relational specification available to a further articulation. Temporal persistence would require an independently established physical ordering.

The Persistent Consequence Principle is a kinematic bridge: it specifies what remains available after an admitted articulation without asserting physical realization. In particular,

$$R_{\mathrm{sp}}(G) \nRightarrow \tau(e) \neq \varnothing. \tag{57}$$

A counterstructure is immediate. Let

$$\mathfrak{C}_{\text{transient}} = \{G, e\} \tag{58}$$

contain the primitive ground and one completed articulation, but no retained structural consequence:

$$\tau(e) = \varnothing. \tag{59}$$

This structure satisfies the self-presentation assumption while containing no basis for relational extension. Therefore, occurrence does not entail persistence.

3.2 Persistence Does Not Yet Establish Provenance

Even when

$$\tau(e) \neq \varnothing, \tag{60}$$

the trace need not encode its direction of production. In particular,

$$\tau(e) \neq \varnothing \nRightarrow e \triangleleft \tau(e), \tag{61}$$

where $\triangleleft$ would denote an intrinsically directed provenance relation.

A trace might preserve:

1. the fact that a relation was completed;
2. compatibility conditions generated by that completion;
3. an orientation class;
4. an endpoint constraint;
5. or a complete ordered history.

These possibilities contain different amounts of information. Persistence alone does not determine which of them is retained. Accordingly,

$$\text{trace existence} \neq \text{complete history retention}. \tag{62}$$

This distinction will later permit histories to share a causal endpoint without requiring them to be identical in provenance.

3.3 Non-Absolute Position and Recentering

The presenting and presented roles are non-absolute. A position occurring as presented in one articulation is not thereby intrinsically prohibited from functioning as presenting in a subsequent articulation.

This motivates a second bridge.

Recentering Principle. If a relational position remains structurally available after an articulation, it is eligible to serve as the origin of a further articulation whenever doing so is compatible with the established relational constraints.

Schematically,

$$G_{\mathrm{pr}} \longrightarrow G_{\mathrm{pd}} \longrightarrow G_{\mathrm{pd,pr}}, \tag{63}$$

where $G_{\mathrm{pd,pr}}$ denotes the same primitive occupant occurring first in a presented role and subsequently in a presenting role.

Equation (63) should not be interpreted as motion through prior space or evolution through prior time. It represents compositional dependence: the second articulation uses a position made available by the first.

Recentering is permissive rather than closure-generating: it makes an available position eligible for further articulation but does not admit every recentering or guarantee that both orientations remain available after every history. Those conditions require Repeatable-Extension Closure.

3.4 Mirror Exchange Versus Extension

The mirror operation J defined in Eq. (46) satisfies

$$J^2 = I. \tag{64}$$

Repeated application therefore produces only the two-element orbit

$$G_{\mathrm{pr}} \overset{J}{\leftrightarrow} G_{\mathrm{pd}}. \tag{65}$$

Mirror exchange alone does not generate an extended relational history. Persistent extension requires an operation that appends another articulated orientation while retaining the previously established composition. The two orientation labels used for this purpose will be denoted by

$$B = \{0, 1\}. \tag{66}$$

The labels 0 and 1 encode the two relational orientations of the presenting–presented articulation. They do not denote two primitive substances.

Unlike mirror exchange, repeated extension is not assumed to return a history to its initial state. The distinction is therefore

$$\boxed{\begin{array}{c} J: \text{ exchange of relational role,} \\ b \in B: \text{ orientation of a persistent extension.} \end{array}} \tag{67}$$

The existence of two orientation labels does not yet establish that either label may be appended after every admitted history. In particular, persistence, recentering, and two orientation labels do not imply closure under all finite extension sequences. The unrestricted binary history space used below therefore requires an additional bridge principle.

3.5 The Binary History Monoid

The history construction adopts the following kinematic closure condition.

Repeatable-Extension Closure Principle. The empty history ϵ is kinematically admitted. For every kinematically admitted finite history w and every orientation $b \in B = \{0, 1\}$, the extended history wb is also kinematically admitted.

This principle is stronger than the Persistent Consequence and Recentering Principles. Persistence retains a consequence, and recentering permits an available position to serve in a further articulation. Repeatable-extension closure additionally stipulates that both extension orientations remain kinematically available after every admitted finite history.

Let $\mathcal{H}$ be the smallest set of finite histories satisfying this closure condition. Equivalently,

$$\mathcal{H} = B^* = \{0, 1\}^*, \tag{68}$$

including the empty word ϵ.

Only after $\mathcal{H}$ has been defined do the right-extension maps take the form

$$E_b : \mathcal{H} \longrightarrow \mathcal{H}, \qquad E_b(w) = wb, \qquad b \in B. \tag{69}$$

Thus, explicitly,

$$E_0(w) = w0, \qquad E_1(w) = w1. \tag{70}$$

These maps are total precisely because repeatable-extension closure makes wb an element of $\mathcal{H}$ for every $w \in \mathcal{H}$ and every $b \in B$.

The extension maps are not involutions:

$$E_0^n \neq I_{\mathcal{H}}, \qquad E_1^n \neq I_{\mathcal{H}} \qquad \text{for every } n > 0. \tag{71}$$

For example,

$$E_0^n(\epsilon) = \underbrace{00\cdots0}_{n \text{ symbols}}. \tag{72}$$

Their nonperiodic action distinguishes persistent extension from the mirror involution J.

For histories

$$w = b_1 b_2 \cdots b_m \qquad \text{and} \qquad z = c_1 c_2 \cdots c_n, \tag{73}$$

define composition by concatenation:

$$w \cdot z = b_1 b_2 \cdots b_m c_1 c_2 \cdots c_n. \tag{74}$$

The empty word satisfies

$$\epsilon \cdot w = w \cdot \epsilon = w, \tag{75}$$

and concatenation is associative:

$$(w \cdot z) \cdot y = w \cdot (z \cdot y). \tag{76}$$

Therefore,

$$\big(\mathcal{H}, \cdot, \epsilon\big) \tag{77}$$

is the free monoid generated by 0 and 1.

Proposition 3.1 (History-space construction)**.** *Subject to Minimal Self-Presentation, persistent consequence, recentering, the two-orientation articulation, and the Repeatable-Extension Closure Principle, the kinematically admitted finite histories are exactly the elements of*

$$\mathcal{H} = \{0, 1\}^*. \tag{78}$$

Every admitted finite extension sequence is represented uniquely by one word in $\mathcal{H}$, and every word in $\mathcal{H}$ represents one kinematically admitted finite extension sequence.

Proof. The empty sequence is represented by ϵ. Suppose an admitted sequence of n extensions is represented by

$$w = b_1 \cdots b_n. \tag{79}$$

For either $b_{n+1} \in B$, the Repeatable-Extension Closure Principle admits

$$wb_{n+1}. \tag{80}$$

Induction therefore admits every finite word over B.

Conversely, the admitted history set is defined as the smallest set containing ϵ and closed under $w \mapsto wb$ for $b \in B$. It consequently contains no histories other than finite words generated by these extensions. Because B^* is free, two extension sequences determine the same word only when they contain the same symbols in the same order. Hence the admitted history set is exactly B^*, with the stated unique representation. □

The proposition characterizes kinematically admitted histories; it does not supply a dynamics selecting which histories are physically realized.

3.6 Length, Counts, and Provenance

For a history

$$w = b_1 b_2 \cdots b_n, \tag{81}$$

define its length by

$$|w| = n. \tag{82}$$

Define the orientation counts

$$N_0(w) = \#\{i : b_i = 0\}, \qquad N_1(w) = \#\{i : b_i = 1\}. \tag{83}$$

Then

$$|w| = N_0(w) + N_1(w). \tag{84}$$

Counts are additive under composition:

$$N_b(wz) = N_b(w) + N_b(z), \qquad b \in \{0, 1\}. \tag{85}$$

The complete word w retains ordered provenance. By contrast, the pair

$$\big(N_0(w), N_1(w)\big) \tag{86}$$

retains only the number of occurrences of each orientation.

For example,

$$w = 0011 \tag{87}$$

and

$$w' = 0101 \tag{88}$$

satisfy

$$\big(N_0(w), N_1(w)\big) = \big(N_0(w'), N_1(w')\big) = (2, 2), \tag{89}$$

while

$$w \neq w'. \tag{90}$$

This distinction will later support endpoint convergence without complete history identification.

Let

$$h(w) \tag{91}$$

denote retained provenance and let

$$\kappa(w) \tag{92}$$

denote any surviving internal constraint not captured by endpoint counts. A complete relational state may then be represented schematically as

$$\Psi(w) = \Big(w, h(w), \kappa(w)\Big). \tag{93}$$

No particular physical interpretation of h or κ is assumed. They are included to prevent an endpoint quotient from being mistaken for total erasure of relational information.

3.7 Mirror Action on Histories

Extend the primitive mirror exchange to the binary alphabet by

$$J_B(0) = 1, \qquad J_B(1) = 0. \tag{94}$$

Define the induced map

$$J_* : \mathcal{H} \longrightarrow \mathcal{H} \tag{95}$$

by applying J_B positionwise:

$$J_*(b_1 b_2 \cdots b_n) = J_B(b_1) J_B(b_2) \cdots J_B(b_n). \tag{96}$$

For example,

$$J_*(00101) = 11010. \tag{97}$$

The induced mirror satisfies

$$J_*^2 = I_{\mathcal{H}}, \tag{98}$$

and preserves history length:

$$|J_*(w)| = |w|. \tag{99}$$

It exchanges the orientation counts:

$$N_0(J_*w) = N_1(w), \qquad N_1(J_*w) = N_0(w). \tag{100}$$

It is also compatible with concatenation:

$$J_*(wz) = J_*(w)J_*(z). \tag{101}$$

Thus J_* is an involutive automorphism of the history monoid.

The history pair

$$\{w, J_*(w)\} \tag{102}$$

is globally invariant under mirror exchange:

$$J_*\big(\{w, J_*(w)\}\big) = \{w, J_*(w)\}. \tag{103}$$

This provides a precise form of global exchange invariance compatible with locally distinct histories.

3.8 Primitive-Ground Projection

Every history remains an articulation of the same primitive occupant. Define

$$\Pi_G : \mathcal{H} \longrightarrow \{G\} \tag{104}$$

by

$$\Pi_G(w) = G \qquad \text{for every } w \in \mathcal{H}. \tag{105}$$

The projection satisfies

$$\Pi_G(wz) = \Pi_G(w) = \Pi_G(z) = G. \tag{106}$$

Equation (105) expresses common primitive grounding; it does not imply that the histories are identical:

$$\Pi_G(w) = \Pi_G(z) \not\Rightarrow w = z. \tag{107}$$

The relational history space therefore combines global primitive-ground invariance with local relational-history differentiation.

The resulting history space is

$$\mathcal{H} = B^* = \{0, 1\}^*,$$

with ordered provenance, mirror symmetry, and common primitive grounding. The next section introduces endpoint recombination and identifies the corresponding quotient structure.

4 Endpoint Recombination and the Quotient Structure

The preceding section constructed an ordered history space while preserving the provenance of each relational extension. The present section asks under what condition distinct histories may nevertheless determine the same effective relational endpoint. We introduce an endpoint map, prove that its fibers define a composition-compatible equivalence relation, and show that the resulting quotient is isomorphic to the free commutative monoid $\mathbb{N}^2$. This recombination concerns endpoint equivalence only: it does not identify the underlying histories, erase their ordering at the history level, or yet assign a causal interpretation to the quotient.

4.1 Histories and Relational Endpoints

The history space

$$\mathcal{H} = \{0,1\}^* \tag{108}$$

retains the complete ordered sequence of relational extensions. Distinct words therefore represent distinct compositional histories:

$$01 \neq 10, \tag{109}$$

and

$$0011 \neq 0101. \tag{110}$$

A causal pregeometry requires an additional distinction between a complete history and the relational endpoint reached by that history. Different ordered compositions may converge to one endpoint without becoming identical as histories.

Define the endpoint-count map

$$q : \mathcal{H} \longrightarrow \mathbb{N}^2 \tag{111}$$

by

$$q(w) = \big(N_0(w), N_1(w)\big). \tag{112}$$

For example,

$$q(01) = q(10) = (1,1), \tag{113}$$

and

$$q(0011) = q(0101) = q(1001) = (2,2). \tag{114}$$

The equality of endpoint data does not imply equality of histories:

$$q(w) = q(w') \nRightarrow w = w'. \tag{115}$$

4.2 Endpoint-Recombination Principle

The identification of histories at the endpoint level requires an explicit bridge principle.

Endpoint-Recombination Principle. Two relational histories determine the same endpoint position when they contain the same number of 0- and 1-oriented extensions:

$$w \sim_q w' \quad \Longleftrightarrow \quad N_0(w) = N_0(w') \ \wedge \ N_1(w) = N_1(w'). \tag{116}$$

Equivalently,

$$w \sim_q w' \quad \Longleftrightarrow \quad q(w) = q(w'). \tag{117}$$

The principle asserts endpoint equivalence, not total relational identity. It permits

$$w \sim_q w' \tag{118}$$

together with

$$h(w) \neq h(w') \tag{119}$$

or

$$\kappa(w) \neq \kappa(w'). \tag{120}$$

The Endpoint-Recombination Principle does not follow from binary articulation alone. Without it, all finite histories remain distinct and form a binary tree. It is therefore classified as a compositional bridge principle.

4.3 Endpoint Equivalence Is an Equivalence Relation

Proposition 4.1. *The relation $\sim_q$ defined in Eq.* (116) *is an equivalence relation on $\mathcal{H}$.*

Proof. For every $w \in \mathcal{H}$,

$$N_0(w) = N_0(w), \qquad N_1(w) = N_1(w), \tag{121}$$

so $w \sim_q w$. Hence $\sim_q$ is reflexive.

If $w \sim_q w'$, then

$$N_0(w) = N_0(w'), \qquad N_1(w) = N_1(w'). \tag{122}$$

By symmetry of equality,

$$N_0(w') = N_0(w), \qquad N_1(w') = N_1(w), \tag{123}$$

so $w' \sim_q w$. Hence $\sim_q$ is symmetric.

Finally, suppose

$$w \sim_q w' \qquad \text{and} \qquad w' \sim_q w''. \tag{124}$$

Then

$$N_b(w) = N_b(w') = N_b(w'') \qquad \text{for } b \in \{0, 1\}. \tag{125}$$

Therefore $w \sim_q w''$, and $\sim_q$ is transitive. □

For $w \in \mathcal{H}$, denote its endpoint-equivalence class by

$$[w]_q = \{w' \in \mathcal{H} : w' \sim_q w\}. \tag{126}$$

The endpoint quotient is

$$\mathcal{E} = \mathcal{H}/\sim_q. \tag{127}$$

4.4 Compatibility with Relational Composition

For the quotient to support relational composition, endpoint equivalence must be preserved under concatenation.

Proposition 4.2 (Congruence of endpoint equivalence)**.** *The equivalence relation $\sim_q$ is a monoid congruence on $\mathcal{H}$. If*

$$w \sim_q w' \qquad \textit{and} \qquad z \sim_q z', \tag{128}$$

then

$$wz \sim_q w'z'. \tag{129}$$

Proof. Because orientation counts are additive under concatenation,

$$N_b(wz) = N_b(w) + N_b(z) \tag{130}$$

for $b \in \{0, 1\}$.

The assumptions imply

$$N_b(w) = N_b(w') \tag{131}$$

and

$$N_b(z) = N_b(z'). \tag{132}$$

Therefore,

$$\begin{aligned} N_b(wz) &= N_b(w) + N_b(z) && (133)\\ &= N_b(w') + N_b(z') && (134)\\ &= N_b(w'z'). && (135) \end{aligned}$$

This holds for both $b = 0$ and $b = 1$, so

$$wz \sim_q w'z'. \tag{136}$$

□

Proposition 4.2 permits composition on $\mathcal{E}$ to be defined by

$$[w]_q \oplus [z]_q = [wz]_q. \tag{137}$$

The operation is well-defined because changing representatives does not change the resulting equivalence class.

The class of the empty history,

$$[\epsilon]_q, \tag{138}$$

is the identity:

$$[\epsilon]_q \oplus [w]_q = [w]_q \oplus [\epsilon]_q = [w]_q. \tag{139}$$

Associativity follows from associativity of history concatenation. Hence

$$\big(\mathcal{E}, \oplus, [\epsilon]_q\big) \tag{140}$$

is a monoid.

4.5 Commutativity at the Endpoint Level

History concatenation is not commutative:

$$wz \neq zw \tag{141}$$

in general. Endpoint composition is commutative.

Proposition 4.3. *For every $w, z \in \mathcal{H}$,*

$$[wz]_q = [zw]_q. \tag{142}$$

Therefore,

$$[w]_q \oplus [z]_q = [z]_q \oplus [w]_q. \tag{143}$$

Proof. For $b \in \{0, 1\}$,

$$\begin{aligned} N_b(wz) &= N_b(w) + N_b(z) && (144)\\ &= N_b(z) + N_b(w) && (145)\\ &= N_b(zw). && (146) \end{aligned}$$

Therefore $wz \sim_q zw$, and their equivalence classes are equal. □

In particular,

$$[01]_q = [10]_q. \tag{147}$$

Equation (147) represents path convergence at the endpoint level. It does not assert that the two histories have identical provenance.

4.6 Characterization by Adjacent Exchanges

The count-equivalence relation can also be generated locally.

Define the elementary adjacent exchange

$$01 \longleftrightarrow 10. \tag{148}$$

Proposition 4.4. *Two histories satisfy $w \sim_q w'$ if and only if one can be transformed into the other by a finite sequence of adjacent exchanges $01 \leftrightarrow 10$.*

Proof. An adjacent exchange preserves both N_0 and N_1. Hence any finite sequence of such exchanges preserves endpoint equivalence.

Conversely, suppose $w \sim_q w'$. The two words contain the same multiset of symbols. By repeatedly exchanging adjacent unequal symbols, each word can be rearranged into the canonical form

$$0^{N_0(w)}1^{N_1(w)}. \tag{149}$$

Since w and w' have equal counts, they have the same canonical form. Reversing the exchanges used for w' transforms the canonical form into w'. Thus w can be transformed into w' through finitely many adjacent exchanges. □

This result shows that the full endpoint quotient is generated by the elementary recombination relation

$$01 \sim_q 10. \tag{150}$$

The physical or structural justification of this local exchange remains a separate question. In the present construction, it is the minimal local relation generating count-based endpoint recombination.

4.7 The Endpoint-Monoid Isomorphism

Define

$$\overline{q} : \mathcal{E} \longrightarrow \mathbb{N}^2 \tag{151}$$

by

$$\overline{q}([w]_q) = \big(N_0(w), N_1(w)\big). \tag{152}$$

Theorem 4.1 (Endpoint-monoid theorem)**.** *The map $\overline{q}$ is a monoid isomorphism:*

$$\big(\mathcal{E}, \oplus, [\epsilon]_q\big) \cong \big(\mathbb{N}^2, +, (0,0)\big). \tag{153}$$

Proof. The map is well-defined because equivalent histories have equal counts.

It is injective because

$$\overline{q}([w]_q) = \overline{q}([z]_q) \tag{154}$$

implies

$$N_0(w) = N_0(z), \qquad N_1(w) = N_1(z), \tag{155}$$

and hence

$$[w]_q = [z]_q. \tag{156}$$

It is surjective because, for every $(u, v) \in \mathbb{N}^2$, the history

$$0^u 1^v \tag{157}$$

satisfies

$$\overline{q}\big([0^u 1^v]_q\big) = (u, v). \tag{158}$$

Finally, it preserves composition:

$$\begin{aligned}\overline{q}\big([w]_q \oplus [z]_q\big) &= \overline{q}([wz]_q) && (159)\\ &= \big(N_0(wz), N_1(wz)\big) && (160)\\ &= \big(N_0(w) + N_0(z), N_1(w) + N_1(z)\big) && (161)\\ &= \overline{q}([w]_q) + \overline{q}([z]_q). && (162)\end{aligned}$$

The identity class maps to $(0,0)$. Therefore $\overline{q}$ is a monoid isomorphism. □

The quotient

$$\mathcal{H}/\!\sim_q \tag{163}$$

is therefore the free commutative monoid on the two endpoint generators

$$e_0 = (1,0), \qquad e_1 = (0,1). \tag{164}$$

4.8 Endpoint Extension Maps

The history extensions E_0 and E_1 descend to maps on endpoint space:

$$\overline{E}_0, \overline{E}_1 : \mathbb{N}^2 \longrightarrow \mathbb{N}^2, \tag{165}$$

where

$$\overline{E}_0(u,v) = (u+1, v), \tag{166}$$

and

$$\overline{E}_1(u,v) = (u, v+1). \tag{167}$$

These maps commute:

$$\overline{E}_0 \circ \overline{E}_1 = \overline{E}_1 \circ \overline{E}_0. \tag{168}$$

They remain independent in the monoid sense. No positive power of one equals a positive power of the other:

$$\overline{E}_0^{\,m} \neq \overline{E}_1^{\,n} \qquad \text{for all } m, n > 0. \tag{169}$$

Indeed,

$$\overline{E}_0^{\,m}(0,0) = (m,0), \tag{170}$$

whereas

$$\overline{E}_1^{\,n}(0,0) = (0,n). \tag{171}$$

The endpoint structure therefore has two independent, commuting, indefinitely iterable extension generators.

This independence is a property of the selected endpoint quotient. It is not derived from PGND alone.

4.9 History Fibers

For an endpoint $(u, v) \in \mathbb{N}^2$, define its history fiber by

$$\mathcal{F}_{u,v} = q^{-1}(u, v). \tag{172}$$

Every history in $\mathcal{F}_{u,v}$ has length

$$u + v \tag{173}$$

and contains exactly u occurrences of 0 and v occurrences of 1.

Proposition 4.5 (Fiber cardinality)**.** *The number of histories converging to* (u, v) *is*

$$|\mathcal{F}_{u,v}| = \binom{u+v}{u} = \binom{u+v}{v}. \tag{174}$$

Proof. A history in $\mathcal{F}_{u,v}$ is a word of length $u + v$ containing u symbols 0 and v symbols 1. Choosing the u positions occupied by 0 uniquely determines the word. There are

$$\binom{u+v}{u} \tag{175}$$

such choices. □

The multiplicity in Eq. (174) measures the number of ordered relational histories associated with one endpoint. It should not yet be interpreted as probability, amplitude, entropy, or path-integral weight.

4.10 Mirror Exchange on Endpoint Space

The history involution J_* induces an endpoint involution

$$\overline{J} : \mathbb{N}^2 \longrightarrow \mathbb{N}^2 \tag{176}$$

defined by

$$\overline{J}(u, v) = (v, u). \tag{177}$$

The endpoint mirror satisfies

$$\overline{J}^{\,2} = I_{\mathbb{N}^2} \tag{178}$$

and exchanges the endpoint generators:

$$\overline{J}(e_0) = e_1, \qquad \overline{J}(e_1) = e_0. \tag{179}$$

The maps q, J_*, and $\overline{J}$ satisfy

$$q \circ J_* = \overline{J} \circ q. \tag{180}$$

Thus the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{J_*} & \mathcal{H} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{N}^2 & \xrightarrow[\overline{J}]{} & \mathbb{N}^2. \end{array} \tag{181}$$

Mirror exchange therefore survives endpoint projection as coordinate exchange.

4.11 Common Primitive Ground

Define

$$\Pi_{\mathcal{E}} : \mathcal{E} \longrightarrow \{G\} \tag{182}$$

by

$$\Pi_{\mathcal{E}}([w]_q) = G. \tag{183}$$

This is well-defined because every history has the same primitive-ground projection:

$$\Pi_G(w) = G. \tag{184}$$

The ground projections satisfy

$$\Pi_G = \Pi_{\mathcal{E}} \circ q. \tag{185}$$

The complete architecture at this stage is therefore

$$\begin{array}{ccccc} G & \xrightarrow{R_{\mathrm{sp}}} & \mathcal{H} & \xrightarrow{q} & \mathcal{E} \cong \mathbb{N}^2 \\ & & & \searrow^{\Pi_G} & \downarrow{\scriptstyle \Pi_{\mathcal{E}}} \\ & & & & \{G\}. \end{array} \tag{186}$$

Relational endpoint multiplicity therefore does not entail primitive-ground multiplicity.

4.12 Alternative Quotients

The count quotient is not uniquely selected by the existence of binary histories. Two alternative constructions demonstrate this.

No endpoint recombination. Let endpoint identity be history identity:

$$w \sim_{\mathrm{tree}} w' \quad \Longleftrightarrow \quad w = w'. \tag{187}$$

Then

$$\mathcal{H}/\!\sim_{\mathrm{tree}} \cong \mathcal{H}, \tag{188}$$

and the number of endpoints at depth n is

$$2^n. \tag{189}$$

The resulting structure is a binary tree.

Length-only recombination. Define

$$w \sim_\ell w' \quad \Longleftrightarrow \quad |w| = |w'|. \tag{190}$$

Then every history of length n determines one endpoint, and

$$\mathcal{H}/\!\sim_\ell \cong \mathbb{N}. \tag{191}$$

The resulting endpoint structure is a one-generator chain.

The three quotient choices therefore give:

$$\begin{aligned} \sim_{\text{tree}} \quad &\longrightarrow \quad \text{binary tree}, \\ \sim_q \quad &\longrightarrow \quad \mathbb{N}^2, \\ \sim_\ell \quad &\longrightarrow \quad \mathbb{N}. \end{aligned} \tag{192}$$

This demonstrates that PGND-compatible binary articulation does not uniquely determine the count quotient. Endpoint recombination is the additional condition responsible for the two-generator structure.

4.13 Status of the Endpoint Construction

Table 2 summarizes the logical status of the endpoint construction and separates its derived results from its bridge principles and physical interpretations.

Table 2: Logical status of the endpoint construction

Structure or result	**Logical status**
Binary history space $\mathcal{H}$	Conditional on persistent consequence, recentering, two-orientation articulation, and repeatable-extension closure
Endpoint-count map q	Definition
Count-based endpoint equivalence	Bridge principle
Equivalence-relation properties	Derived
Monoid congruence	Derived
Endpoint commutativity	Derived from count equivalence
$\mathcal{E} \cong \mathbb{N}^2$	Theorem
Independent endpoint generators	Derived within the selected quotient
History-fiber multiplicity	Derived combinatorial result
Mirror equivariance	Derived
Common primitive-ground projection	PGND-compatibility condition
Physical causal interpretation	Not yet introduced
Lorentzian metric interpretation	Not yet introduced
Unique selection of $\sim_q$	Not derived

The next section derives an order on $\mathcal{E}$ directly from relational extension and tests whether the resulting ordered structure satisfies the defining properties required of a discrete causal order.

5 Relational Reachability and Candidate Causal Order

The endpoint quotient established in the preceding section determines which relational histories recombine into the same effective state, but it does not yet determine how different endpoint states are ordered. This section defines relational reachability by the possibility of further endpoint extension and proves that the resulting relation is a locally finite partial order on $\mathbb{N}^2$. We characterize its covers, intervals, antichains, chains, and past and future sets, thereby obtaining a candidate causal architecture from relational composition. The mathematical order follows from the endpoint construction; its interpretation as physical causality requires the separately stated causal-specialization principle.

5.1 Order from Relational Extension

The endpoint space

$$\mathcal{E} \cong \big(\mathbb{N}^2, +\big) \tag{193}$$

carries a natural order induced by compositional extension.

For

$$p, q \in \mathcal{E}, \tag{194}$$

define

$$p \preceq_R q \quad \Longleftrightarrow \quad \exists r \in \mathcal{E} \text{ such that } p + r = q. \tag{195}$$

The subscript R indicates relational or compositional reachability. At this stage, $\preceq_R$ is not yet assumed to be physical causal order.

Let

$$p = (u, v), \qquad q = (u', v'). \tag{196}$$

Proposition 5.1 (Coordinate characterization)**.** *The relational reachability order satisfies*

$$(u, v) \preceq_R (u', v') \quad \Longleftrightarrow \quad u \le u' \ \wedge \ v \le v'. \tag{197}$$

Proof. Suppose

$$(u, v) \preceq_R (u', v'). \tag{198}$$

Then there exists

$$(a, b) \in \mathbb{N}^2 \tag{199}$$

such that

$$(u, v) + (a, b) = (u', v'). \tag{200}$$

Therefore

$$u' = u + a, \qquad v' = v + b. \tag{201}$$

Since $a, b \geq 0$, it follows that

$$u \leq u', \qquad v \leq v'. \tag{202}$$

Conversely, suppose

$$u \leq u', \qquad v \leq v'. \tag{203}$$

Then

$$r = (u' - u, v' - v) \in \mathbb{N}^2 \tag{204}$$

and

$$(u, v) + r = (u', v'). \tag{205}$$

Hence

$$(u, v) \preceq_R (u', v'). \tag{206}$$

□

Thus the order is not imposed independently upon endpoint space. It is obtained from the existence of an admissible extension connecting one endpoint to another.

5.2 Partial-Order Properties

Theorem 5.1 (Relational endpoint order)**.** *The relation $\preceq_R$ is a partial order on $\mathcal{E}$.*

Proof. For reflexivity, let $p \in \mathcal{E}$. Since

$$p + (0, 0) = p, \tag{207}$$

we have

$$p \preceq_R p. \tag{208}$$

For antisymmetry, suppose

$$p \preceq_R q \qquad \text{and} \qquad q \preceq_R p. \tag{209}$$

Writing

$$p = (u, v), \qquad q = (u', v'), \tag{210}$$

Proposition 5.1 gives

$$u \leq u', \qquad v \leq v', \tag{211}$$

and

$$u' \leq u, \qquad v' \leq v. \tag{212}$$

Therefore

$$u = u', \qquad v = v', \tag{213}$$

so $p = q$.

For transitivity, suppose

$$p \preceq_R q \qquad \text{and} \qquad q \preceq_R s. \tag{214}$$

Then there exist $r_1, r_2 \in \mathcal{E}$ such that

$$p + r_1 = q \tag{215}$$

and

$$q + r_2 = s. \tag{216}$$

Consequently,

$$p + (r_1 + r_2) = s. \tag{217}$$

Since $r_1 + r_2 \in \mathcal{E}$, it follows that

$$p \preceq_R s. \tag{218}$$

□

Define the associated strict order by

$$p \prec_R q \quad \iff \quad p \preceq_R q \ \wedge \ p \neq q. \tag{219}$$

The strict order is irreflexive and transitive. It contains no directed cycles.

Corollary 5.1 (Acyclicity)**.** *There exists no finite sequence*

$$p_0 \prec_R p_1 \prec_R \cdots \prec_R p_n \prec_R p_0. \tag{220}$$

Proof. If such a sequence existed, transitivity would imply

$$p_0 \prec_R p_0, \tag{221}$$

contradicting irreflexivity. □

Acyclicity here follows from the monotone extension structure of $\mathbb{N}^2$. It should not yet be interpreted as an independently derived arrow of physical time.

5.3 Cover Relations and Local Extension

For $p, q \in \mathcal{E}$, say that q covers p, written

$$p \prec\!\!\cdot_R q, \tag{222}$$

when

$$p \prec_R q \tag{223}$$

and there exists no $z \in \mathcal{E}$ satisfying

$$p \prec_R z \prec_R q. \tag{224}$$

Proposition 5.2 (Characterization of covers)**.** *For $p = (u, v)$, the only elements covering p are*

$$(u + 1, v) \qquad \textit{and} \qquad (u, v + 1). \tag{225}$$

Proof. Both points in Eq. (225) are strict extensions of p. No integer-coordinate point lies strictly between p and either extension.

Conversely, suppose

$$q = (u + a, v + b) \tag{226}$$

covers p, where $a, b \in \mathbb{N}$ and $a + b > 0$. If $a + b > 1$, then at least one of a or b is positive and a one-generator extension of p lies strictly between p and q. This contradicts the assumption that q covers p. Therefore $a + b = 1$, giving either

$$(a, b) = (1, 0) \tag{227}$$

or

$$(a, b) = (0, 1). \tag{228}$$

□

The Hasse graph of $(\mathcal{E}, \preceq_R)$ has exactly two outgoing covers at every endpoint:

$$(u, v) \lessdot_R (u + 1, v), \tag{229}$$

and

$$(u, v) \lessdot_R (u, v + 1). \tag{230}$$

Consequently,

$$\#\,\mathrm{Cov}^+(p) = 2, \tag{231}$$

where $\mathrm{Cov}^+(p)$ denotes the set of future covers of p.

Each endpoint has at most two immediate predecessors:

$$(u - 1, v), \qquad (u, v - 1), \tag{232}$$

whenever the corresponding coordinates remain nonnegative. Every interior endpoint of $\mathbb{N}^2$ has exactly two, while boundary endpoints have fewer.

These results establish a finite order-theoretic cover neighborhood. They do not by themselves determine which reachability relations, if any, represent elementary physical propagation. That interpretation requires an additional bridge principle.

Local-Propagation Principle. Only cover relations are interpreted as elementary propagation steps:

$$p \longrightarrow_{\mathrm{loc}} q \quad \Longleftrightarrow \quad p \lessdot_R q. \tag{233}$$

Every non-cover reachability relation is interpreted as a composite of elementary cover steps.

Conditional on this principle, every endpoint has exactly two elementary future propagation channels:

$$\#\mathcal{N}^{+}_{\mathrm{loc}}(p) = 2. \tag{234}$$

The distinction is therefore as follows: The finite cover neighborhood is a derived order-theoretic result; the interpretation of covers as elementary propagation steps is supplied by the Local-Propagation Principle; and finite local propagation is a conditional consequence of combining the two.

5.4 Local Finiteness

For comparable endpoints $p \preceq_R q$, define the closed order interval

$$I[p,q] = \{z \in \mathcal{E} : p \preceq_R z \preceq_R q\}. \tag{235}$$

Let

$$p = (u_0, v_0), \qquad q = (u_1, v_1), \tag{236}$$

with

$$\Delta u = u_1 - u_0, \qquad \Delta v = v_1 - v_0. \tag{237}$$

Proposition 5.3 (Interval cardinality)**.** *If $p \preceq_R q$, then*

$$|I[p,q]| = (\Delta u + 1)(\Delta v + 1). \tag{238}$$

Proof. An endpoint $z = (u, v)$ belongs to $I[p,q]$ exactly when

$$u_0 \le u \le u_1 \tag{239}$$

and

$$v_0 \le v \le v_1. \tag{240}$$

There are $\Delta u + 1$ permitted values of u and $\Delta v + 1$ permitted values of v. Their choices are independent, giving

$$|I[p,q]| = (\Delta u + 1)(\Delta v + 1). \tag{241}$$

□

Every interval is finite. Hence:

Corollary 5.2. *The ordered set*

$$(\mathcal{E}, \preceq_R) \tag{242}$$

is locally finite.

The endpoint structure is therefore a locally finite partially ordered set.

5.5 Comparable and Incomparable Endpoints

Two endpoints $p = (u, v)$ and $q = (u', v')$ are comparable when either

$$u \leq u' \ \wedge \ v \leq v' \tag{243}$$

or

$$u' \leq u \ \wedge \ v' \leq v. \tag{244}$$

They are incomparable precisely when their coordinate differences have opposite signs:

$$(u' - u)(v' - v) < 0. \tag{245}$$

For example,

$$(2, 0) \qquad \text{and} \qquad (0, 2) \tag{246}$$

are incomparable. Neither can be obtained from the other by adding a nonnegative endpoint extension.

In a later causal interpretation, comparable pairs will be assigned timelike or null separation, while incomparable pairs will be assigned spacelike separation. At the present stage, incomparability means only absence of directed relational reachability.

5.6 Generation Rank and Antichains

Define the generation-rank function

$$\rho : \mathcal{E} \longrightarrow \mathbb{N} \tag{247}$$

by

$$\rho(u, v) = u + v. \tag{248}$$

Every cover relation increases rank by one:

$$p \lessdot_R q \Longrightarrow \rho(q) = \rho(p) + 1. \tag{249}$$

For $n \in \mathbb{N}$, define the rank-n slice

$$\Sigma_n = \{(u, v) \in \mathbb{N}^2 : u + v = n\}. \tag{250}$$

The slice contains

$$|\Sigma_n| = n + 1 \tag{251}$$

endpoints:

$$(0, n), (1, n - 1), \ldots, (n - 1, 1), (n, 0). \tag{252}$$

Proposition 5.4. *Each Σ_n is an antichain: no two distinct elements of Σ_n are comparable under $\preceq_R$.*

Proof. Let

$$p = (u, n - u) \tag{253}$$

and

$$q = (u', n - u') \tag{254}$$

be distinct elements of Σ_n. If $u < u'$, then

$$n - u > n - u'. \tag{255}$$

Thus one coordinate increases while the other decreases, so p and q are incomparable. The case $u' > u$ is symmetric. □

The rank function gives a structural foliation of the endpoint poset by antichains. It is not yet physical time. Interpreting ρ or a rescaling of it as time is a later physical specialization.

5.7 Chains and Relational Path Length

Let

$$p = (u_0, v_0) \preceq_R q = (u_1, v_1). \tag{256}$$

Every saturated chain from p to q consists of

$$\Delta u \tag{257}$$

extensions by e_0 and

$$\Delta v \tag{258}$$

extensions by e_1. Therefore every saturated chain has

$$L_R(p, q) = \Delta u + \Delta v \tag{259}$$

cover steps.

The number of saturated chains is

$$\mathcal{N}_{\text{chain}}(p, q) = \binom{\Delta u + \Delta v}{\Delta u} = \binom{\Delta u + \Delta v}{\Delta v}. \tag{260}$$

This is the interval analogue of the history-fiber cardinality. Distinct ordered extension histories can converge to the same endpoint q.

Equation (259) should not yet be interpreted as proper time. In the Minkowski specialization below, proper-time separation will depend on the product $\Delta u \Delta v$, not merely on the total number of cover steps.

5.8 Past and Future Sets

For $p \in \mathcal{E}$, define its relational future and past by

$$J_R^+(p) = \{q \in \mathcal{E} : p \preceq_R q\}, \tag{261}$$

and

$$J_R^-(p) = \{q \in \mathcal{E} : q \preceq_R p\}. \tag{262}$$

For $p = (u_0, v_0)$,

$$J_R^+(p) = \{(u, v) : u \geq u_0,\ v \geq v_0\}, \tag{263}$$

and

$$J_R^-(p) = \{(u, v) : 0 \leq u \leq u_0,\ 0 \leq v \leq v_0\}. \tag{264}$$

The mirror involution exchanges the endpoint coordinates but preserves the order:

$$p \preceq_R q \quad \Longleftrightarrow \quad \overline{J}(p) \preceq_R \overline{J}(q). \tag{265}$$

Thus

$$\overline{J} \in \mathrm{Aut}\big(\mathcal{E}, \preceq_R\big). \tag{266}$$

5.9 Causal-Specialization Principle

The preceding results establish a locally finite compositional partial order. Conditional on the Local-Propagation Principle, its cover relations also define a finite elementary propagation graph. Neither result by itself establishes that relational reachability represents physical causal influence.

The required physical interpretation is therefore stated separately.

Causal-Specialization Principle. Relational reachability from p to q is interpreted as proto-causal accessibility when:

1. q is reachable from p through a finite sequence of elementary cover extensions selected by the Local-Propagation Principle;
2. the resulting directed relation is compatible with the established acyclic order; and
3. later relational consequences at q may depend upon structure available at p.

Under this specialization, define

$$p \preceq_C q \quad \Longleftrightarrow \quad p \preceq_R q. \tag{267}$$

Equation (267) assigns a proto-causal interpretation to the already established reachability order. It is not a new order-theoretic theorem. The mathematical properties of $\preceq_C$ are inherited from $\preceq_R$, while its interpretation as physical accessibility remains conditional on the Local-Propagation and Causal-Specialization Principles.

$$(\mathcal{E}, \preceq_C) \tag{268}$$

is a locally finite partially ordered set.

The resulting structure satisfies the basic mathematical definition of a causal set: it is a locally finite partially ordered set [2, 3]. Under the Causal-Specialization Principle, its order is interpreted as proto-causal accessibility. Its physical adequacy remains conditional upon the continuum and symmetry tests developed below.

5.10 What Has and Has Not Been Established

The construction has established:

1. an order derived from endpoint extension;
2. a coordinatewise characterization of that order;
3. reflexivity, antisymmetry, and transitivity;
4. acyclicity of the strict order;
5. a finite order-theoretic cover neighborhood;
6. conditional on the Local-Propagation Principle, a finite elementary propagation neighborhood;
7. local finiteness of all causal intervals;
8. antichain slices of fixed generation rank;
9. split–recombine causal-diamond structure.

The finite order-theoretic cover neighborhood is derived from the product order on $\mathbb{N}^2$. Its interpretation as a finite elementary propagation neighborhood depends on the independent Local-Propagation Principle.

It has not yet established:

1. a physical temporal coordinate;
2. a metric or proper-time function;
3. null, timelike, or spacelike metric classification;
4. continuum topology;
5. Lorentz invariance;
6. curvature;
7. physical dynamics;
8. empirical adequacy.

The next section introduces a coordinate embedding into discrete $1+1$-dimensional Minkowski spacetime. That embedding will determine whether the relational order coincides with Minkowski causal order and whether the two elementary extension classes define null boundaries.

6 Discrete Minkowski Embedding

The preceding section produced a locally finite candidate causal order from relational reachability. This section determines whether that order admits a concrete Lorentzian representation by embedding its endpoint coordinates into $1+1$-dimensional Minkowski spacetime. We show that the two elementary extensions become future-directed null displacements, that relational reachability agrees exactly with the induced Minkowski causal order, and that endpoint intervals possess a calculable proper-time relation. This embedding establishes an explicit Lorentzian specialization of the relational structure; it does not show that Minkowski geometry, its metric scale, or its physical interpretation follows from PGND alone.

6.1 From Endpoint Coordinates to Minkowski Coordinates

Let

$$\ell > 0 \tag{269}$$

denote a fundamental relational length scale. Define the map

$$\iota_\ell : \mathcal{E} \longrightarrow \mathbb{R}^{1,1} \tag{270}$$

by

$$\iota_\ell(u, v) = (T, X), \tag{271}$$

where

$$T = \ell(u + v), \qquad X = \ell(v - u). \tag{272}$$

The inverse coordinate relations on the image are

$$u = \frac{T - X}{2\ell}, \qquad v = \frac{T + X}{2\ell}. \tag{273}$$

Because $u, v \in \mathbb{N}$, the image of ι_ℓ is the discrete future wedge

$$\mathbb{M}^{1,1}_{\ell,+} = \left\{ (T, X) : T \geq |X|,\ \frac{T - X}{2\ell} \in \mathbb{N},\ \frac{T + X}{2\ell} \in \mathbb{N} \right\}. \tag{274}$$

Equivalently, T/ℓ and X/ℓ are integers of the same parity, with $T \geq |X|$. The endpoint mirror operation

$$\overline{J}(u, v) = (v, u) \tag{275}$$

is represented under ι_ℓ by

$$(T, X) \longmapsto (T, -X). \tag{276}$$

Thus mirror exchange preserves T while reversing the sign of the emergent spatial coordinate. This representation should not be projected backward onto the primitive level: $\overline{J}$ becomes spatial reflection only after the Minkowski specialization has been introduced.

6.2 Minkowski Causal Order

Equip $\mathbb{R}^{1,1}$ with the Minkowski metric

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{277}$$

so that

$$ds^2 = -dT^2 + dX^2. \tag{278}$$

For two points

$$P = (T, X), \qquad Q = (T', X'), \tag{279}$$

define the future-directed Minkowski causal order by

$$P \preceq_M Q \quad \Longleftrightarrow \quad \Delta T \geq |\Delta X|, \tag{280}$$

where

$$\Delta T = T' - T, \qquad \Delta X = X' - X. \tag{281}$$

The inequality in Eq. (280) implies $\Delta T \geq 0$.

Theorem 6.1 (Order-isomorphic Minkowski embedding)**.** *For all $p, q \in \mathcal{E}$,*

$$p \preceq_R q \quad \Longleftrightarrow \quad \iota_\ell(p) \preceq_M \iota_\ell(q). \tag{282}$$

Consequently,

$$\iota_\ell : (\mathcal{E}, \preceq_R) \longrightarrow \left(\mathbb{M}^{1,1}_{\ell,+}, \preceq_M\right) \tag{283}$$

is an order isomorphism onto its image.

Proof. Let

$$p = (u, v), \qquad q = (u', v'), \tag{284}$$

and define

$$\Delta u = u' - u, \qquad \Delta v = v' - v. \tag{285}$$

Under ι_ℓ,

$$\Delta T = \ell(\Delta u + \Delta v), \tag{286}$$

and

$$\Delta X = \ell(\Delta v - \Delta u). \tag{287}$$

Suppose $p \preceq_R q$. By Proposition 5.1,

$$\Delta u \geq 0, \qquad \Delta v \geq 0. \tag{288}$$

Therefore

$$\Delta u + \Delta v \geq |\Delta v - \Delta u|, \tag{289}$$

which gives

$$\Delta T \geq |\Delta X|. \tag{290}$$

Hence

$$\iota_\ell(p) \preceq_M \iota_\ell(q). \tag{291}$$

Conversely, suppose

$$\Delta T \geq |\Delta X|. \tag{292}$$

Using the inverse coordinate relations,

$$\Delta u = \frac{\Delta T - \Delta X}{2\ell}, \tag{293}$$

and

$$\Delta v = \frac{\Delta T + \Delta X}{2\ell}. \tag{294}$$

The causal inequality implies

$$\Delta T - \Delta X \geq 0, \qquad \Delta T + \Delta X \geq 0. \tag{295}$$

Thus

$$\Delta u \geq 0, \qquad \Delta v \geq 0, \tag{296}$$

so $p \preceq_R q$. □

The order constructed from relational extension therefore coincides exactly with the restriction of $1+1$-dimensional Minkowski causal order to the discrete image lattice.

6.3 Null Coordinates

Define continuous null coordinates

$$U = \ell u, \qquad V = \ell v. \tag{297}$$

Then

$$T = U + V, \qquad X = V - U. \tag{298}$$

Differentiating gives

$$dT = dU + dV, \qquad dX = dV - dU. \tag{299}$$

Substitution into Eq. (278) yields

$$ds^2 = -(dU + dV)^2 + (dV - dU)^2 \tag{300}$$
$$= -4\, dU\, dV. \tag{301}$$

Thus u and v, after multiplication by ℓ, function as discrete null coordinates.

This result depends on the Minkowski specialization. The primitive labels 0 and 1 are not intrinsically null directions. They acquire that interpretation through the embedding ι_ℓ.

6.4 Elementary Extensions Are Null

The two elementary endpoint extensions are

$$\overline{E}_0(u, v) = (u + 1, v) \tag{302}$$

and

$$\overline{E}_1(u, v) = (u, v + 1). \tag{303}$$

Under ι_ℓ, an E_0-extension gives

$$(\Delta T, \Delta X) = (\ell, -\ell), \tag{304}$$

while an E_1-extension gives

$$(\Delta T, \Delta X) = (\ell, +\ell). \tag{305}$$

For either extension,

$$\Delta s^2 = -(\Delta T)^2 + (\Delta X)^2 = -\ell^2 + \ell^2 = 0. \tag{306}$$

Therefore, the cover relations of the endpoint poset are represented by null steps:

$$\begin{aligned} &\overline{E}_0 \longleftrightarrow \text{left-directed null extension}, \\ &\overline{E}_1 \longleftrightarrow \text{right-directed null extension}. \end{aligned} \tag{307}$$

The two uninterrupted orientation histories

$$0^n \qquad \text{and} \qquad 1^n \tag{308}$$

generate the two null boundaries:

$$X = -T \qquad \text{and} \qquad X = +T. \tag{309}$$

Mixed histories lie inside the causal cone.

6.5 Lorentzian Classification of Endpoint Separation

Let

$$p = (u, v), \qquad q = (u', v'), \tag{310}$$

and define

$$\Delta s^2 = -(\Delta T)^2 + (\Delta X)^2. \tag{311}$$

Using Eqs. (286) and (287),

$$\Delta s^2 = -\ell^2(\Delta u + \Delta v)^2 + \ell^2(\Delta v - \Delta u)^2 \tag{312}$$

$$= -4\ell^2 \Delta u \Delta v. \tag{313}$$

The sign of $\Delta u \Delta v$ determines the causal classification.

Timelike separation. If

$$\Delta u > 0, \qquad \Delta v > 0, \tag{314}$$

then

$$\Delta s^2 < 0. \tag{315}$$

The endpoints are strictly timelike related.

Null separation. If

$$\Delta u \geq 0, \qquad \Delta v \geq 0, \tag{316}$$

and exactly one of $\Delta u, \Delta v$ vanishes, then

$$\Delta s^2 = 0. \tag{317}$$

The endpoints are null related.

Spacelike separation. If

$$\Delta u \Delta v < 0, \tag{318}$$

then

$$\Delta s^2 > 0. \tag{319}$$

The endpoints are incomparable under $\preceq_R$ and spacelike separated under the Minkowski embedding.

Thus:

$$\boxed{\begin{aligned} &p \prec_R q,\ \Delta u \Delta v > 0 \iff \text{timelike}, \\ &p \prec_R q,\ \Delta u \Delta v = 0 \iff \text{null}, \\ &p \parallel_R q \iff \text{spacelike}, \end{aligned}} \tag{320}$$

where $p \parallel_R q$ denotes order incomparability.

6.6 Proper-Time Function

For causally related endpoints, define the Minkowski proper-time separation by

$$\tau(p,q) = \sqrt{(\Delta T)^2 - (\Delta X)^2}. \tag{321}$$

Using Eq. (313),

$$\tau^2(p,q) = 4\ell^2 \Delta u \Delta v. \tag{322}$$

Hence

$$\tau(p,q) = 2\ell\sqrt{\Delta u \Delta v}. \tag{323}$$

This differs from the saturated-chain length:

$$L_R(p,q) = \Delta u + \Delta v. \tag{324}$$

The distinction is important. Relational generation depth and Minkowski proper time are different quantities:

$$\boxed{L_R(p,q) \neq \frac{\tau(p,q)}{\ell} \quad \text{in general.}} \tag{325}$$

For a fixed ΔT, proper time is maximal when

$$\Delta u = \Delta v, \tag{326}$$

corresponding to $\Delta X = 0$. It vanishes on the null boundaries, where one coordinate increment is zero.

6.7 Causal Diamonds

For causally related continuum events, the intersection of the causal future of the earlier event with the causal past of the later event is an Alexandrov interval, also called a causal diamond [10].

For $p \preceq_R q$, the order interval

$$I[p,q] = J_R^+(p) \cap J_R^-(q) \tag{327}$$

is mapped by ι_ℓ to the discrete points in the Minkowski causal diamond

$$J_M^+\big(\iota_\ell(p)\big) \cap J_M^-\big(\iota_\ell(q)\big). \tag{328}$$

For the elementary recombination

$$(u,v) \longrightarrow \begin{cases} (u+1,v), \\ (u,v+1), \end{cases} \quad \longrightarrow (u+1,v+1), \tag{329}$$

the two intermediate endpoints are incomparable, while the final endpoint lies in the common future of both.

Under ι_ℓ, the structure becomes

$$(T,X) \longrightarrow \begin{cases} (T+\ell, X-\ell), \\ (T+\ell, X+\ell), \end{cases} \quad \longrightarrow (T+2\ell, X). \tag{330}$$

This is the minimal split–recombine causal diamond.

6.8 Interval Cardinality and Proper-Time Scaling

From Proposition 5.3,

$$|I[p,q]| = (\Delta u + 1)(\Delta v + 1). \tag{331}$$

Expanding,

$$|I[p,q]| = \Delta u \Delta v + \Delta u + \Delta v + 1. \tag{332}$$

Using

$$\Delta u \Delta v = \frac{\tau^2}{4\ell^2} \tag{333}$$

and

$$\Delta u + \Delta v = \frac{\Delta T}{\ell}, \tag{334}$$

one obtains the exact relation

$$|I[p,q]| = \frac{\tau^2(p,q)}{4\ell^2} + \frac{\Delta T}{\ell} + 1. \tag{335}$$

For a family of timelike intervals in which both

$$\Delta u \to \infty \qquad \text{and} \qquad \Delta v \to \infty \tag{336}$$

with their ratio bounded away from 0 and ∞, the quadratic term dominates:

$$|I[p,q]| \sim \frac{\tau^2(p,q)}{4\ell^2}. \tag{337}$$

Thus the causal-interval cardinality has the growth law

$$\boxed{|I[p,q]| \propto \tau^2,} \tag{338}$$

which is characteristic of a two-dimensional causal geometry.

The qualification excluding asymptotically null intervals is necessary. Along a null boundary,

$$\tau = 0, \tag{339}$$

while the number of lattice points in the interval may still grow linearly with ΔT.

6.9 Logical Status of the Embedding

The embedding establishes the conditional result

$$\boxed{\begin{array}{l}\text{count-based endpoint recombination} + \text{extension-induced partial order} \\ + \text{Minkowski specialization} \\ \Longrightarrow \text{a discrete } 1+1\text{-dimensional Lorentzian causal structure.}\end{array}} \tag{340}$$

More specifically, the construction provides an exact order isomorphism with a discrete Minkowski future wedge, two elementary null extension classes, timelike, null, and spacelike relational separation, discrete causal diamonds, a proper-time function, and quadratic causal-interval growth.

The metric in Eq. (278) is a physical specialization compatible with the relational order. The order isomorphism shows that this specialization is mathematically faithful, but it does not show that the Minkowski metric is uniquely forced by PGND. Nor does the regular lattice possess exact continuous Lorentz symmetry, because a generic Lorentz boost need not preserve $\mathbb{M}^{1,1}_{\ell,+}$. Thus,

$$\text{Lorentzian causal order} \not\Rightarrow \text{exact microscopic Lorentz symmetry.} \tag{341}$$

The continuum analysis in the next section determines what is recovered under refinement and what symmetry questions remain unresolved.

7 Continuum Scaling and Emergent Lorentzian Structure

The preceding section established an exact correspondence between relational reachability and the causal order induced on a discrete Minkowski null lattice. Exact order compatibility at a fixed lattice scale, however, does not by itself establish the recovery of continuum geometry or Lorentz symmetry. This section therefore studies a refining family of relational lattices, defines a counting-based volume measure, and proves convergence to the Alexandrov-interval scaling of $1+1$-dimensional Minkowski spacetime. The result supplies a calculable continuum limit while preserving an important distinction: the limiting causal geometry is Lorentz covariant, whereas each regular finite-scale lattice retains a preferred discrete frame.

7.1 A Family of Relational Lattices

A continuum limit cannot be defined by considering a single lattice with fixed relational scale. Introduce a family

$$\{\mathcal{E}_\ell\}_{\ell>0}\,, \tag{342}$$

where

$$\mathcal{E}_\ell = \mathbb{N}^2 \tag{343}$$

as an abstract ordered set, while its Minkowski realization depends upon ℓ:

$$\iota_\ell(u,v) = \big(\ell(u+v), \ell(v-u)\big). \tag{344}$$

The embedded lattice is

$$\Lambda_\ell = \iota_\ell\big(\mathcal{E}_\ell\big) \subset \mathbb{R}^{1,1}. \tag{345}$$

As $\ell \to 0$, the number of relational endpoints representing any fixed finite continuum region increases. The continuum limit considered here holds the physical coordinates

$$T = \ell(u + v), \qquad X = \ell(v - u) \tag{346}$$

finite while u and v become large.

The limit therefore does not represent an endpoint evolving toward a continuum. It compares a family of increasingly fine discrete realizations of the same candidate continuum region.

7.2 Density in the Continuum Future Wedge

Define the continuum future wedge

$$W^+ = \left\{(T, X) \in \mathbb{R}^{1,1} : T \geq |X|,\ T \geq 0\right\}. \tag{347}$$

Proposition 7.1 (Continuum density)**.** *For every point*

$$P = (T, X) \in W^+ \tag{348}$$

and every sequence $\ell_k \to 0$, there exists a sequence

$$P_k \in \Lambda_{\ell_k} \tag{349}$$

such that

$$P_k \longrightarrow P. \tag{350}$$

Proof. Define the continuum null coordinates

$$U = \frac{T - X}{2}, \qquad V = \frac{T + X}{2}. \tag{351}$$

Since $P \in W^+$,

$$U \geq 0, \qquad V \geq 0. \tag{352}$$

For each ℓ_k, choose

$$u_k = \left\lfloor \frac{U}{\ell_k} \right\rfloor \tag{353}$$

and

$$v_k = \left\lfloor \frac{V}{\ell_k} \right\rfloor. \tag{354}$$

Let

$$P_k = \iota_{\ell_k}(u_k, v_k). \tag{355}$$

Then

$$\ell_k u_k \longrightarrow U, \qquad \ell_k v_k \longrightarrow V. \tag{356}$$

Consequently,

$$T_k = \ell_k(u_k + v_k) \longrightarrow U + V = T, \tag{357}$$

and

$$X_k = \ell_k(v_k - u_k) \longrightarrow V - U = X. \tag{358}$$

□

Thus the union of increasingly fine embedded endpoint lattices becomes dense in W^+.

This is a density result, not yet a complete theorem of Lorentzian manifold convergence. It does not by itself establish convergence of curvature, dynamics, quantum fields, or global topology.

7.3 Order Compatibility Across Scales

For every fixed $\ell > 0$, Theorem 6.1 gives

$$p \preceq_R q \quad \Longleftrightarrow \quad \iota_\ell(p) \preceq_M \iota_\ell(q). \tag{359}$$

The causal-order correspondence is therefore exact at every lattice scale. It is not merely an asymptotic property.

For continuum points, convergence of order is clearest away from the null boundary.

Proposition 7.2 (Stability of strict causal type)**.** *Let*

$$P, Q \in W^+ \tag{360}$$

have strictly timelike separation:

$$\Delta T > |\Delta X|. \tag{361}$$

If

$$P_k, Q_k \in \Lambda_{\ell_k} \tag{362}$$

converge respectively to P, Q, then, for sufficiently large k,

$$P_k \prec_M Q_k. \tag{363}$$

Likewise, if

$$|\Delta X| > |\Delta T|, \tag{364}$$

then sufficiently accurate lattice approximants remain spacelike and incomparable.

Proof. The functions

$$F_\pm(P, Q) = \Delta T \pm \Delta X \tag{365}$$

are continuous in the endpoint coordinates.

For strictly timelike separation,

$$F_+(P, Q) > 0, \qquad F_-(P, Q) > 0. \tag{366}$$

Continuity implies that both inequalities remain positive for all sufficiently close approximants. Hence the approximating pair is timelike related.

For strictly spacelike separation, F_+ and F_- have opposite signs. Their signs likewise remain unchanged under sufficiently small perturbations. □

Null separation is more delicate because it lies on the boundary between causal and spacelike relations. A sequence converging to a null-related pair may approach from the timelike or spacelike side unless the lattice approximation is chosen to preserve one null coordinate exactly. The lattice itself nevertheless contains exact null chains generated by E_0 and E_1.

7.4 Relational Counting Measure

The map

$$(u, v) \longmapsto (T, X) \tag{367}$$

has Jacobian

$$\frac{\partial(T, X)}{\partial(u, v)} = \begin{pmatrix} \ell & \ell \\ -\ell & \ell \end{pmatrix}, \tag{368}$$

with determinant

$$\left| \det \frac{\partial(T, X)}{\partial(u, v)} \right| = 2\ell^2. \tag{369}$$

Accordingly, assign each endpoint the continuum coordinate-volume weight

$$\nu_\ell = 2\ell^2. \tag{370}$$

For a finite subset

$$A \subset \mathcal{E}_\ell, \tag{371}$$

define the rescaled counting measure

$$\mu_\ell(A) = 2\ell^2 |A|. \tag{372}$$

The factor $2\ell^2$ is not derived from PGND. It is selected by the Jacobian of the Minkowski embedding so that relational counting can be compared with continuum coordinate volume.

7.5 Continuum Volume of a Causal Interval

Let $P, Q \in W^+$ be fixed strictly timelike-related continuum events. For each sufficiently small $\ell > 0$, choose lattice approximants

$$p_\ell, q_\ell \in \Lambda_\ell \tag{373}$$

such that

$$p_\ell \longrightarrow P, \qquad q_\ell \longrightarrow Q \qquad \text{as } \ell \to 0. \tag{374}$$

Because P and Q are strictly timelike related, the approximants may be chosen so that

$$p_\ell \preceq_R q_\ell \tag{375}$$

for all sufficiently small ℓ.

Write their integer null coordinates as

$$p_\ell = (u_{0,\ell}, v_{0,\ell}), \qquad q_\ell = (u_{1,\ell}, v_{1,\ell}), \tag{376}$$

and define

$$\Delta u_\ell = u_{1,\ell} - u_{0,\ell}, \qquad \Delta v_\ell = v_{1,\ell} - v_{0,\ell}. \tag{377}$$

The corresponding discrete interval contains

$$\big| I[p_\ell, q_\ell] \big| = (\Delta u_\ell + 1)(\Delta v_\ell + 1) \tag{378}$$

endpoints. Therefore,

$$\mu_\ell\big(I[p_\ell, q_\ell]\big) = 2\ell^2(\Delta u_\ell + 1)(\Delta v_\ell + 1) \tag{379}$$

$$= 2\ell^2 \Delta u_\ell \Delta v_\ell + 2\ell^2(\Delta u_\ell + \Delta v_\ell) + 2\ell^2. \tag{380}$$

Let τ_ℓ and ΔT_ℓ denote, respectively, the proper-time and coordinate-time separations of the embedded approximants. Then

$$\tau_\ell^2 = 4\ell^2 \Delta u_\ell \Delta v_\ell \tag{381}$$

and

$$\Delta T_\ell = \ell(\Delta u_\ell + \Delta v_\ell). \tag{382}$$

Hence, at every lattice scale,

$$\boxed{\mu_\ell\big(I[p_\ell, q_\ell]\big) = \frac{\tau_\ell^2}{2} + 2\ell \Delta T_\ell + 2\ell^2.} \tag{383}$$

Let τ denote the proper-time separation between P and Q. The convergence in Eq. (374) implies

$$\tau_\ell \longrightarrow \tau, \qquad \Delta T_\ell \longrightarrow \Delta T. \tag{384}$$

In particular, ΔT_ℓ remains bounded, and therefore

$$2\ell \Delta T_\ell + 2\ell^2 \longrightarrow 0. \tag{385}$$

It follows that

$$\boxed{\mu_\ell\big(I[p_\ell, q_\ell]\big) \longrightarrow \frac{\tau^2}{2}.} \tag{386}$$

In $1+1$-dimensional Minkowski spacetime, the coordinate area of an Alexandrov interval of proper-time duration τ is [10]

$$V_{\mathrm{Alex}}(P, Q) = \frac{\tau^2}{2}. \tag{387}$$

Consequently,

$$\boxed{\lim_{\ell \to 0} \mu_\ell\big(I[p_\ell, q_\ell]\big) = V_{\mathrm{Alex}}(P, Q).} \tag{388}$$

Theorem 7.1 (Causal-volume convergence)**.** *Let $P, Q \in W^+$ be fixed strictly timelike-related continuum events, and let $p_\ell, q_\ell \in \Lambda_\ell$ be causally ordered lattice approximants converging respectively to P and Q. Then the rescaled endpoint-counting measure satisfies*

$$\lim_{\ell \to 0} \mu_\ell\big(I[p_\ell, q_\ell]\big) = V_{\mathrm{Alex}}(P, Q). \tag{389}$$

The theorem establishes convergence of causal-interval volume under the stated lattice-approximation procedure. It does not establish convergence of every geometric or dynamical observable.

7.6 Myrheim–Meyer Dimension Comparison

A standard order-theoretic diagnostic in causal-set theory is the Myrheim–Meyer dimension estimator, which compares the ordering fraction of a finite causal set with its expected continuum value [11, 12, 13]. Although the endpoint structure constructed here is a deterministic regular lattice rather than a Poisson sprinkling, its interval ordering fraction can be calculated exactly and compared asymptotically with this estimator.

Let

$$a = \Delta u + 1, \qquad b = \Delta v + 1, \qquad N = |I[p,q]| = ab. \tag{390}$$

The number R of strict order relations between distinct elements of the interval is

$$\begin{aligned} R &= \sum_{\substack{0 \le i < a,\ 0 \le j < b \\ (i,j) \neq (0,0)}} (a-i)(b-j) \\ &= \frac{a(a+1)b(b+1)}{4} - ab. \end{aligned} \tag{391}$$

The corresponding ordering fraction is therefore

$$r(a,b) = \frac{2R}{N(N-1)} = \frac{(a+1)(b+1)-4}{2(ab-1)}. \tag{392}$$

For any family of nondegenerate intervals in which $a, b \to \infty$ with their ratio bounded away from zero and infinity,

$$\lim_{a,b\to\infty} r(a,b) = \frac{1}{2}. \tag{393}$$

For an Alexandrov interval in d-dimensional Minkowski spacetime, the Myrheim–Meyer comparison function is

$$f(d) = \frac{\Gamma(d+1)\Gamma(d/2)}{2\Gamma(3d/2)}. \tag{394}$$

Since

$$f(2) = \frac{1}{2}, \tag{395}$$

the asymptotic ordering fraction of the endpoint intervals agrees with the $1+1$-dimensional Myrheim–Meyer value. Thus,

$$d_{\mathrm{MM}} \longrightarrow 2 \qquad \text{as} \qquad a, b \longrightarrow \infty. \tag{396}$$

This agreement supplies an order-theoretic comparison with a standard causal-set dimension diagnostic, independently of the explicit null-coordinate embedding. It does not imply that the endpoint lattice is statistically equivalent to a Poisson sprinkling. Finite intervals retain regular-lattice and boundary corrections, including additional comparable pairs lying on common lattice-null lines, but these contributions are subleading in the stated large-interval limit.

7.7 Continuum Lorentz Transformations

A proper, orthochronous Lorentz boost of rapidity η acts on Minkowski coordinates by

$$\begin{pmatrix} T' \\ X' \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} T \\ X \end{pmatrix}. \tag{397}$$

In null coordinates,

$$U = \frac{T-X}{2}, \qquad V = \frac{T+X}{2}, \tag{398}$$

the boost acts as

$$U' = e^{\eta}U, \qquad V' = e^{-\eta}V. \tag{399}$$

The product is invariant:

$$U'V' = UV. \tag{400}$$

Therefore,

$$ds'^2 = -4\,dU'\,dV' = -4\,dU\,dV = ds^2. \tag{401}$$

The continuum target geometry is thus exactly Lorentz invariant.

The discrete lattice is not. A generic boost does not map

$$\Lambda_\ell \tag{402}$$

onto itself because $e^{\eta}u$ and $e^{-\eta}v$ need not be integers.

This is the familiar difficulty that a regular spacetime lattice selects preferred directions, whereas Lorentz-invariant Poisson sprinkling avoids selecting a measurable preferred frame [14, 15]. The present regular lattice therefore does not possess exact microscopic Lorentz invariance.

Hence:

$$\mathrm{Aut}\big(\Lambda_\ell, \preceq_M\big) \neq SO^+(1,1) \tag{403}$$

for fixed ℓ.

7.8 Vanishing Frame-Dependent Counting Correction

A generic Lorentz boost does not map Λ_ℓ to itself. Consequently, a comparison between the interval counts assigned in two frames requires an explicit lattice-approximation prescription.

Let $P, Q \in W^+$ be fixed strictly timelike-related continuum events, and let

$$p_\ell, q_\ell \in \Lambda_\ell \tag{404}$$

be lattice approximants satisfying

$$p_\ell \longrightarrow P, \qquad q_\ell \longrightarrow Q \qquad \text{as } \ell \to 0. \tag{405}$$

For example, in null coordinates one may select lattice coordinates by rounding each continuum null coordinate downward to the nearest multiple of ℓ. Because

P and Q are strictly timelike related, the approximants remain causally ordered for all sufficiently small ℓ.

Let B_η be a fixed Lorentz boost of rapidity η, and choose separate lattice approximants

$$p'_\ell, q'_\ell \in \Lambda_\ell \tag{406}$$

such that

$$p'_\ell \longrightarrow B_\eta P, \qquad q'_\ell \longrightarrow B_\eta Q \qquad \text{as } \ell \to 0. \tag{407}$$

Write the corresponding null-coordinate separations as

$$\Delta U_\ell = \ell \Delta u_\ell, \qquad \Delta V_\ell = \ell \Delta v_\ell, \tag{408}$$

and

$$\Delta U'_\ell = \ell \Delta u'_\ell, \qquad \Delta V'_\ell = \ell \Delta v'_\ell. \tag{409}$$

Under the stated approximation prescription,

$$\Delta U_\ell = \Delta U + O(\ell), \qquad \Delta V_\ell = \Delta V + O(\ell), \tag{410}$$

while

$$\Delta U'_\ell = e^{\eta} \Delta U + O(\ell), \qquad \Delta V'_\ell = e^{-\eta} \Delta V + O(\ell). \tag{411}$$

The rescaled interval counts are

$$\mu_\ell\big(I[p_\ell, q_\ell]\big) = 2\ell^2 \big(\Delta u_\ell + 1\big)\big(\Delta v_\ell + 1\big) \tag{412}$$

and

$$\mu_\ell\big(I[p'_\ell, q'_\ell]\big) = 2\ell^2 \big(\Delta u'_\ell + 1\big)\big(\Delta v'_\ell + 1\big). \tag{413}$$

Equivalently,

$$\mu_\ell\big(I[p_\ell, q_\ell]\big) = 2\big(\Delta U_\ell + \ell\big)\big(\Delta V_\ell + \ell\big), \tag{414}$$

with the analogous expression for the primed approximants.

Equations (410) and (411) imply

$$\mu_\ell\big(I[p_\ell, q_\ell]\big) = 2\Delta U \Delta V + O(\ell) \tag{415}$$

and

$$\mu_\ell\big(I[p'_\ell, q'_\ell]\big) = 2\big(e^{\eta}\Delta U\big)\big(e^{-\eta}\Delta V\big) + O(\ell). \tag{416}$$

Since

$$2\big(e^{\eta}\Delta U\big)\big(e^{-\eta}\Delta V\big) = 2\Delta U \Delta V = \frac{\tau^2}{2}, \tag{417}$$

both sequences converge to the same Lorentz-invariant Alexandrov volume:

$$\lim_{\ell \to 0} \mu_\ell\big(I[p_\ell, q_\ell]\big) = \lim_{\ell \to 0} \mu_\ell\big(I[p'_\ell, q'_\ell]\big) = \frac{\tau^2}{2}. \tag{418}$$

Consequently,

$$\lim_{\ell \to 0} \Big[\mu_\ell\big(I[p'_\ell, q'_\ell]\big) - \mu_\ell\big(I[p_\ell, q_\ell]\big)\Big] = 0. \tag{419}$$

For fixed nonzero τ, the relative errors also vanish:

$$\frac{\mu_\ell\big(I[p_\ell, q_\ell]\big) - \tau^2/2}{\tau^2/2} \longrightarrow 0, \tag{420}$$

and likewise for the primed sequence.

Thus, under the stated approximation prescription, the interval-volume observable has a Lorentz-invariant continuum limit even though the microscopic lattice retains a preferred frame. This is a controlled asymptotic counting result; it does not establish exact microscopic Lorentz invariance or imply that every observable or possible dynamics has a Lorentz-covariant continuum limit.

7.9 Required Symmetry Criterion

A full emergent-Lorentz result would require a family of physical observables

$$\mathcal{O}_\ell \tag{421}$$

and limiting observables

$$\mathcal{O} \tag{422}$$

such that:

1. $$\mathcal{O}_\ell \longrightarrow \mathcal{O} \qquad \text{as } \ell \to 0; \tag{423}$$

2. the limiting observable depends only on Lorentz-invariant or Lorentz-covariant continuum data;

3. lattice anisotropies vanish relative to the continuum scale;

4. and any continuum dynamics transforms covariantly under

$$SO^+(1,1). \tag{424}$$

The present paper proves this behavior only for the rescaled causal-interval volume. No field equation or dynamical propagator is yet defined.

7.10 Scope of the Continuum Result

The continuum analysis establishes:

1. density of the endpoint lattices in the continuum future wedge;

2. exact causal-order compatibility at every scale;

3. convergence of rescaled endpoint counting to Minkowski Alexandrov volume;

4. asymptotic agreement with the 1+1-dimensional Myrheim–Meyer ordering fraction;

5. exact Lorentz invariance of the continuum target metric;

6. and vanishing frame dependence of the leading interval-volume correction.

It does not establish:

1. exact Lorentz invariance at finite ℓ;

2. a unique continuum topology derived from PGND;

3. convergence of physical fields or dynamics;

4. curvature or gravitational equations;

5. quantum amplitudes or interference;

6. a $3+1$-dimensional continuum;

7. or empirical predictions.

The next section develops countermodels and alternative constructions that test which assumptions are responsible for the causal and continuum properties obtained above.

8 Countermodels and Assumption Audit

8.1 Purpose of the Countermodel Analysis

The construction proceeds through logically distinct articulation, persistence, extension, recombination, ordering, and physical-specialization stages. The following countermodels remove or alter individual assumptions to identify the structural contribution of each. The resulting dependencies are collected in Table 3 and in the minimal chain of Subsection 8.17.

Each arrow introduces either a definition, a derived result, a bridge principle, or a physical specialization. The purpose of this section is to determine which conclusions fail when one of the relevant assumptions is removed.

A countermodel is not treated merely as an objection. It identifies the exact structural contribution of a missing premise.

8.2 Primitive Ground Without Articulation

Consider the model

$$\mathcal{M}_0 = \{G\}, \tag{425}$$

with no presenting–presented articulation.

The model satisfies Minimal PGND:

$$G_{\min}. \tag{426}$$

However, it contains no articulated position set A_0, no history space $\mathcal{H}$, and no relational endpoint structure.

Thus:

$$G_{\min} \not\Rightarrow R_{\mathrm{sp}}(G). \tag{427}$$

Minimal Self-Presentation is therefore an additional articulation bridge.

8.3 Self-Presentation Without Persistence

Let

$$\mathcal{M}_1 = \{G, G_{\mathrm{pr}}, G_{\mathrm{pd}}, R_{\mathrm{sp}}\}, \tag{428}$$

with

$$\omega(G_{\mathrm{pr}}) = \omega(G_{\mathrm{pd}}) = G, \tag{429}$$

but suppose

$$\tau(e) = \varnothing \tag{430}$$

for every completed articulation e.

The model contains one same-occupant relational articulation, but no completed relation remains structurally available for a subsequent composition.

Therefore:

$$R_{\mathrm{sp}}(G) \not\Rightarrow \mathcal{H} = \{0,1\}^*. \tag{431}$$

Persistent consequence is therefore required for iterative history construction.

8.4 Mirror Exchange Without Extension

Let

$$A_0 = \{G_{\mathrm{pr}}, G_{\mathrm{pd}}\} \tag{432}$$

and define

$$J(G_{\mathrm{pr}}) = G_{\mathrm{pd}}, \qquad J(G_{\mathrm{pd}}) = G_{\mathrm{pr}}. \tag{433}$$

Then

$$J^2 = I. \tag{434}$$

Suppose no persistent extension maps E_0, E_1 exist. Repeated application of J produces only

$$G_{\mathrm{pr}}, \quad G_{\mathrm{pd}}, \quad G_{\mathrm{pr}}, \quad G_{\mathrm{pd}}, \quad \ldots \tag{435}$$

within the same two-element orbit.

No arbitrarily long relational history is generated. Hence:

$$J^2 = I \not\Rightarrow \text{persistent extension}. \tag{436}$$

Mirror exchange and persistent extension are therefore independent structures.

8.5 Extension Permission Without Repeatable Closure

Let $\mathcal{H}_{\text{adm}}$ contain the empty history and at least one nonempty extension, but suppose it is not closed under both right extensions:

$$\exists w \in \mathcal{H}_{\text{adm}}\, \exists b \in B \quad wb \notin \mathcal{H}_{\text{adm}}. \tag{437}$$

For example, let

$$\mathcal{H}_{\text{adm}} = \{\epsilon, 0, 1, 00\}. \tag{438}$$

This structure permits further articulation after some histories, but it does not admit all finite binary extension sequences. In particular,

$$\mathcal{H}_{\text{adm}} \neq B^*. \tag{439}$$

Therefore,

$$\text{permission for further extension} \not\Rightarrow \text{repeatable-extension closure.} \tag{440}$$

Thus permission for some further extensions does not make E_0 and E_1 total on the admitted history space; unrestricted binary histories require Repeatable-Extension Closure.

8.6 History Space Without Endpoint Recombination

Let every history remain its own endpoint:

$$w \sim_{\text{tree}} w' \quad \Longleftrightarrow \quad w = w'. \tag{441}$$

Then

$$\mathcal{E}_{\text{tree}} = \mathcal{H}/\!\sim_{\text{tree}} \cong \mathcal{H}. \tag{442}$$

Under the prefix order,

$$w \preceq_{\text{prefix}} z \quad \Longleftrightarrow \quad \exists y \in \mathcal{H} \text{ such that } z = wy. \tag{443}$$

At history length n, the number of endpoints is

$$N_{\text{tree}}(n) = 2^n. \tag{444}$$

The structure branches indefinitely without recombination. It is locally finite under the prefix order, but its slice growth is exponential rather than linear.

Thus

$$\text{binary histories} \not\Rightarrow \mathbb{N}^2 \text{ endpoint geometry.} \tag{445}$$

Count-based recombination converts exponential history branching into polynomial endpoint growth.

8.7 Length-Only Recombination

Define

$$w \sim_\ell w' \quad \Longleftrightarrow \quad |w| = |w'|. \tag{446}$$

Every history of length n is then assigned to the same endpoint. The quotient is

$$\mathcal{E}_\ell \cong \mathbb{N}, \tag{447}$$

with composition

$$[m] + [n] = [m + n]. \tag{448}$$

The endpoint structure is a one-generator chain:

$$0 \preceq 1 \preceq 2 \preceq \cdots . \tag{449}$$

This quotient erases the distinction between 0- and 1-oriented extension counts. It therefore cannot supply two independent null-coordinate classes.

Consequently,

$$\text{recombination} \not\Rightarrow \text{two-dimensional endpoint structure.}$$

The dimension of the quotient depends on which invariants recombination preserves.

8.8 A Non-Congruential Recombination Rule

Suppose a relation $\approx$ identifies only the elementary histories

$$01 \approx 10 \tag{450}$$

while leaving all longer histories unrelated unless they are identical.

Then

$$01 \approx 10, \tag{451}$$

but right concatenation by 0 gives

$$010 \not\approx 100. \tag{452}$$

If quotient composition were defined by

$$[w]_\approx \odot [z]_\approx = [wz]_\approx, \tag{453}$$

the result would depend on the chosen representative:

$$[01]_\approx = [10]_\approx, \tag{454}$$

while

$$[01]_\approx \odot [0]_\approx \neq [10]_\approx \odot [0]_\approx. \tag{455}$$

Thus $\approx$ is not a monoid congruence, and composition on the quotient is not well-defined.

This countermodel establishes:

$$\text{local path identification} \not\Rightarrow \text{composable endpoint quotient.} \tag{456}$$

A composable endpoint quotient therefore requires equivalence to be stable under further composition.

8.9 Endpoint Convergence Without History Identity

Let

$$w = 01, \qquad w' = 10. \tag{457}$$

Then

$$q(w) = q(w') = (1, 1). \tag{458}$$

Suppose, however, that the persistent traces preserve ordering:

$$h(01) \neq h(10). \tag{459}$$

The complete relational states are then

$$\Psi(01) = \big((1, 1), h(01), \kappa(01)\big), \tag{460}$$

and

$$\Psi(10) = \big((1, 1), h(10), \kappa(10)\big). \tag{461}$$

These states share a geometric endpoint while remaining distinct in provenance.

Therefore:

$$q(w) = q(w') \not\Rightarrow \Psi(w) = \Psi(w'). \tag{462}$$

Endpoint identification therefore need not erase path-sensitive relational information.

8.10 Dependent Extension Generators

Suppose the two endpoint extensions are not independent. For example, let

$$\overline{E}_1 = \overline{E}_0. \tag{463}$$

Then both symbols produce the same endpoint increment:

$$\overline{E}_0(n) = \overline{E}_1(n) = n + 1. \tag{464}$$

The generated monoid is

$$(\mathbb{N}, +) \tag{465}$$

rather than

$$(\mathbb{N}^2, +). \tag{466}$$

Consequently:

$$\text{two labels} \not\Rightarrow \text{two independent extension directions.} \tag{467}$$

Two-dimensional endpoint structure therefore requires independent generators, not merely two labels.

8.11 Directed Order Without Local Propagation

Let the endpoint set remain

$$\mathbb{N}^2, \tag{468}$$

but declare every map

$$F_k^{(0)}(u,v) = (u+k, v) \tag{469}$$

and

$$F_k^{(1)}(u,v) = (u, v+k), \tag{470}$$

for arbitrary $k \geq 1$, to be an elementary physical propagation step.

The coordinatewise order and its covers remain unchanged, but every endpoint now has infinitely many elementary propagation channels:

$$\#\mathcal{N}^+(u,v) = \infty. \tag{471}$$

Hence

$$\boxed{\text{partial order with finite covers} \not\Rightarrow \text{local physical propagation.}} \tag{472}$$

The Local-Propagation Principle supplies the additional identification of covers with elementary physical steps.

8.12 Dimensional Growth Without Causal Signature

Consider the Euclidean square lattice

$$\mathbb{Z}^2 \tag{473}$$

with the usual graph distance. The number of lattice points in a large disk satisfies

$$N_{\mathrm{E}}(r) \propto r^2. \tag{474}$$

This structure has effective growth dimension two, but it contains no distinguished causal order, no future and past sets, and no null boundary.

Accordingly,

$$N(r) \propto r^2 \not\Rightarrow 1+1\text{-dimensional Lorentzian causal structure.} \tag{475}$$

Lorentzian interpretation additionally requires the causal order and Minkowski specialization.

8.13 Causal Order Without Metric Scale

Let

$$g = -dT^2 + dX^2 \tag{476}$$

be the Minkowski metric. For any smooth positive function

$$\Omega : W^+ \longrightarrow \mathbb{R}_{>0}, \tag{477}$$

define

$$g_\Omega = \Omega^2(T, X) g. \tag{478}$$

The two metrics have the same null cones and causal order:

$$g(v, v) = 0 \quad \Longleftrightarrow \quad g_\Omega(v, v) = 0. \tag{479}$$

However, they generally assign different proper times and volume measures:

$$dV_\Omega = \Omega^2 \, dV_g. \tag{480}$$

Therefore:

$$\text{causal order} \not\Rightarrow \text{unique metric scale.} \tag{481}$$

The weight $2\ell^2$ therefore supplies a volume normalization not fixed by the causal order.

8.14 Lorentzian Form Without Exact Lattice Lorentz Invariance

The embedded endpoint lattice has the exact interval form

$$\Delta s^2 = -(\Delta T)^2 + (\Delta X)^2. \tag{482}$$

Nevertheless, consider a Lorentz boost with nonzero rapidity η:

$$U' = e^\eta U, \qquad V' = e^{-\eta} V. \tag{483}$$

For a generic lattice point,

$$U = \ell u, \qquad V = \ell v, \tag{484}$$

the transformed quantities

$$\frac{U'}{\ell} = e^\eta u, \qquad \frac{V'}{\ell} = e^{-\eta} v \tag{485}$$

are not integers. Hence the boosted point does not generally lie in Λ_ℓ.

Therefore:

$$\text{Minkowski interval form} \not\Rightarrow \text{exact continuous lattice Lorentz symmetry.} \tag{486}$$

Continuum interval-volume covariance is therefore weaker than exact microscopic lattice Lorentz invariance.

8.15 Common Primitive Ground Without Unique Articulation

Let

$$R_{\mathrm{tree}}, \qquad R_{\mathrm{length}}, \qquad R_{\mathrm{count}} \tag{487}$$

denote articulations producing, respectively:

$$\mathcal{E}_{\mathrm{tree}}, \qquad \mathcal{E}_{\ell}, \qquad \mathcal{E}. \tag{488}$$

Each may possess a primitive-ground projection satisfying

$$\Pi_G \circ R_{\alpha}(G) = G. \tag{489}$$

Nevertheless, the resulting endpoint structures are non-isomorphic:

$$\mathcal{E}_{\mathrm{tree}} \ncong \mathcal{E}_{\ell} \ncong \mathcal{E}. \tag{490}$$

Therefore:

$$\text{common PGND grounding} \nRightarrow \text{unique relational articulation.} \tag{491}$$

Common primitive grounding therefore does not select the count-based articulation.

8.16 Countermodel Summary

Table 3 summarizes the countermodel audit by identifying, for each available structure, the stronger conclusion that does not follow from it alone.

Table 3: Summary of the countermodel audit

Available structure	**Conclusion that does not follow**
Minimal PGND	Self-presentation
Self-presentation	Persistent trace
Mirror involution	Relational extension
Permission for further extension	Repeatable-extension closure
Binary histories	Endpoint recombination
Any recombination	Two-generator geometry
Elementary path identification	Composable quotient
Same endpoint	Same complete relational state
Two orientation labels	Independent generators
Partial order with finite covers	Local physical propagation
Quadratic growth	Lorentzian causal signature
Causal order	Unique metric scale
Minkowski interval form	Exact lattice Lorentz invariance
Common primitive ground	Unique articulation

8.17 Minimal Dependency Chain

The strongest dependency chain supported by the construction is:

$$\begin{aligned} & G_{\min} \\ & + \text{Minimal Self-Presentation} \\ & + \text{Persistent Consequence} \\ & + \text{Recentering} \\ & + \text{Two-Orientation Articulation} \\ & + \text{Repeatable-Extension Closure} \\ & \Longrightarrow \mathcal{H} = \{0,1\}^*. \end{aligned} \tag{492}$$

followed by

$$\begin{aligned} & \mathcal{H} + \text{count-based endpoint recombination} \\ & \Longrightarrow \mathcal{E} \cong \mathbb{N}^2, \end{aligned} \tag{493}$$

followed by

$$\begin{aligned} & \mathcal{E} + \text{extension-induced reachability} \\ & \Longrightarrow (\mathcal{E}, \preceq_R) \text{ is a locally finite poset,} \end{aligned} \tag{494}$$

followed by

$$\begin{aligned} & (\mathcal{E}, \preceq_R) \\ & + \text{Local-Propagation Principle} \\ & + \text{Causal-Specialization Principle} \\ & \Longrightarrow \big(\mathcal{E}, \preceq_C, \longrightarrow_{\mathrm{loc}}\big) \text{ is a locally finite candidate causal structure} \\ & \quad \text{with a finite elementary propagation neighborhood,} \end{aligned} \tag{495}$$

and finally

$$\begin{aligned} & \big(\mathcal{E}, \preceq_C, \longrightarrow_{\mathrm{loc}}\big) \\ & + \text{Minkowski specialization} \\ & + \text{uniform counting scale and refinement rule} \\ & \Longrightarrow \text{a discrete } 1+1\text{-dimensional Lorentzian causal pregeometry} \\ & \quad \text{with the stated continuum-volume limit.} \end{aligned} \tag{496}$$

Separately, the large-interval ordering fraction of the endpoint order approaches the $1+1$-dimensional Myrheim–Meyer value.

The next section interprets this conditional construction and identifies its remaining physical limitations.

9 Physical Interpretation and Scope of the Construction

The preceding sections establish a conditional route from same-occupant relational articulation to a locally finite endpoint order with a fixed-scale $1+1$-dimensional Minkowski realization. This section interprets that result, compares it with causal-set and relational approaches, states the central fixed-scale theorem, and identifies the principal unresolved physical questions.

9.1 Interpretive Status of Same-Occupant Articulation

The elementary articulation places one primitive occupant in two non-equivalent relational positions:

$$\omega\big(G_{\mathrm{pr}}\big) = \omega\big(G_{\mathrm{pd}}\big) = G, \qquad G_{\mathrm{pr}} \neq_{\mathrm{role}} G_{\mathrm{pd}}. \tag{497}$$

These positions are not distinct primitive objects and have no intrinsic spatial, temporal, causal, or subject–object interpretation. They acquire such interpretations only through the subsequent history, order, and Minkowski specializations.

Same-occupant articulation is therefore stronger than the identity $G = G$, but compatible with the primitive-description constraint imposed by PGND. It is an explicit bridge principle rather than a consequence of PGND. The additional principles governing persistence, extension, recombination, and physical specialization perform logically separate work, as established by the countermodel audit in Section 8.

9.2 Admissibility, Realization, and Uniqueness

Let $\mathfrak{B}$ denote the complete collection of bridge principles and physical specializations used in the construction. The conditional admissibility result has the form

$$\mathrm{PGND} + \mathfrak{B} \Longrightarrow \exists \mathcal{E}\ \mathrm{Lorentzian}_{1+1}(\mathcal{E}). \tag{498}$$

It establishes the existence of one PGND-compatible Lorentzian causal structure under the stated conditions. It does not follow that

$$\mathrm{Admissible}(\mathcal{E}) \Longrightarrow \mathrm{Realized}(\mathcal{E}), \tag{499}$$

because the construction supplies no dynamics or realization principle. Nor would realization establish that $\mathcal{E}$ is the unique admissible or realized structure. The result therefore demonstrates compatibility, not physical selection, uniqueness, or empirical adequacy.

9.3 Relation to Causal-Set Theory

Under the Causal-Specialization Principle, $(\mathcal{E}, \preceq_C)$ is a locally finite causal partial order and therefore satisfies the basic mathematical definition of a causal

set [2, 3]. Its order supplies causal organization, while interval cardinality and the weight $2\ell^2$ provide discrete volume data whose continuum limit reproduces the volume scaling of $1+1$-dimensional Alexandrov intervals. This reflects the causal-set relation between causal order, conformal structure, and volume information under the appropriate spacetime conditions [4, 5].

The present construction differs from standard causal-set approaches in its aim and scope. It gives a conditional relational route to one particular causal set rather than postulating a causal order or supplying a stochastic growth law. Classical sequential growth, for example, assigns transition probabilities to causal-set formation under covariance and causality conditions [16, 3]; no analogous dynamics or probability measure has been derived here.

A further limitation is that the regular null lattice selects a preferred discrete frame. Although Lorentz transformations preserve the continuum causal order, they do not generally map the lattice to itself. The model therefore recovers continuum Lorentzian order and interval-volume scaling, not exact microscopic Lorentz invariance. Causal-set theory addresses this problem through Poisson sprinkling, whose random discreteness does not select a measurable preferred Lorentz frame [14, 15, 3]. No comparable stochastic realization mechanism presently follows from the PGND construction.

9.4 Relational Rather Than Substantival Geometry

Spatial and temporal coordinates are not assigned to the primitive ground. The endpoint coordinates (u, v) initially count the two modes of relational extension. Only under the Minkowski specialization do they acquire the null-coordinate interpretation

$$U = \ell u, \qquad V = \ell v.$$

The resulting coordinates

$$T = \ell(u + v), \qquad X = \ell(v - u)$$

respectively measure cumulative extension and the imbalance between the two extension modes. Thus T and X are functions on the endpoint structure, not primitive properties of G or coordinates in a presupposed spacetime.

In this limited sense, the construction represents spacetime geometry through stable relational organization rather than an independently imposed background. This is compatible with broad relational motivations in quantum theory and quantum gravity [6, 7], but it does not derive their quantum or dynamical structures: no quantum state space, probability rule, observer dependence, or gravitational dynamics is established here.

9.5 Order, Metric, and Conformal Freedom

The causally interpreted endpoint order determines which events may be causally related, but not an absolute metric scale. For example,

$$g = -dT^2 + dX^2, \qquad g_\Omega = \Omega^2(T, X)g, \qquad \Omega(T, X) > 0,$$

define the same local null cones and causal order.

For the present model, the Minkowski specialization and uniform scale ℓ select a flat metric, while the endpoint weight $2\ell^2$ fixes the discrete volume normalization. These are additional physical inputs rather than consequences of the causal order. The construction therefore obtains an order-compatible flat interval-volume law, but does not derive a variable conformal factor, curvature, or gravitational dynamics. Section 8 gives the corresponding causal-order-without-metric-scale countermodel.

9.6 Theorem-Level Results

The central fixed-scale result is the following.

Theorem 9.1 (Fixed-scale conditional Lorentzian admissibility)**.** *Let G be a primitive ground satisfying Minimal PGND. Assume:*

1. *Minimal Self-Presentation;*
2. *Persistent Consequence;*
3. *Recentering;*
4. *Two-Orientation Articulation;*
5. *Repeatable-Extension Closure;*
6. *count-based endpoint recombination;*
7. *extension-induced reachability;*
8. *the Local-Propagation Principle;*
9. *the Causal-Specialization Principle; and*
10. *a fixed scale $\ell > 0$ with Minkowski specialization*

$$\iota_\ell(u, v) = \big(\ell(u + v), \ell(v - u)\big).$$

Then

$$\mathcal{E} \cong \mathbb{N}^2, \qquad (u, v) \preceq_R (u', v') \quad \Longleftrightarrow \quad u \leq u' \text{ and } v \leq v'.$$

This order is locally finite and has exactly two future covers at each endpoint, which the Local-Propagation Principle identifies as the two elementary future propagation channels. Under the Causal-Specialization Principle,

$$p \preceq_C q \quad \Longleftrightarrow \quad p \preceq_R q,$$

and ι_ℓ is an order isomorphism onto the causally ordered discrete future wedge

$$\mathbb{M}^{1,1}_{\ell,+} = \iota_\ell(\mathcal{E}) \subset \mathbb{R}^{1,1}.$$

Hence, at every fixed $\ell > 0$, the stated principles yield a discrete $1{+}1$-dimensional Lorentzian causal pregeometry. This conclusion does not imply continuum convergence or exact microscopic Lorentz invariance.

Proof. The history-space construction gives $\mathcal{H} = \{0,1\}^*$, and the quotient results of Section 4 give $\mathcal{E} \cong \mathbb{N}^2$. Section 5 establishes the coordinatewise characterization, local finiteness, and the two future covers. The Local-Propagation and Causal-Specialization Principles supply their respective physical interpretations, while the order-isomorphism result of Section 6 identifies the resulting causal order with that induced on $\mathbb{M}^{1,1}_{\ell,+}$. □

Theorem 9.1 concerns a single fixed lattice scale. The separate refinement result, Theorem 7.1, adds a family $\ell \to 0$, the weight $2\ell^2$, and an endpoint-approximation rule to establish convergence of rescaled interval counts to $1+1$-dimensional Minkowski Alexandrov volume. Thus the first theorem provides the discrete order-compatible realization, while the second establishes its continuum volume-scaling limit under additional assumptions.

9.7 What Remains Unresolved

The construction leaves five groups of questions open:

1. *Realization and selection:* what dynamics, if any, makes relational completion actual and selects persistent binary extension, endpoint recombination, and causal specialization;

2. *Dimension and microscopic symmetry:* why the observed macroscopic dimension is $3+1$ and how exact microscopic Lorentz invariance could arise without a preferred discrete frame;

3. *Geometry and gravity:* how curvature, variable metric structure, and gravitational dynamics emerge;

4. *Matter and quantum structure:* how fields, mass, internal symmetries, quantum amplitudes, and the Born rule arise; and

5. *Uniqueness and empirical adequacy:* whether the constructed structure is uniquely selected and whether its extensions reproduce observed physics.

These questions mark the boundary between the present conditional admissibility result and a dynamically complete physical theory. The next section addresses one part of this boundary by testing whether the binary endpoint construction can be generalized to higher-dimensional Lorentzian geometry.

10 Beyond the $1+1$-Dimensional Witness

The binary construction provides an exact $1+1$-dimensional witness. This section tests whether its finite-generator mechanism can reproduce higher-dimensional Lorentzian causal cones and identifies the additional structures that such a generalization would require.

10.1 Finite Alphabets and the Special $1+1$ Case

Replacing the binary alphabet by

$$\Sigma_m = \{a_1, \ldots, a_m\}$$

and quotienting the free history monoid

$$\mathcal{H}_m = \Sigma_m^*$$

by equal generator multiplicities gives

$$\mathcal{E}_m \cong \mathbb{N}^m$$

with the product order. Its continuum count cone is the positive orthant

$$\mathbb{R}_+^m = \left\{ \sum_{i=1}^m \lambda_i e_i : \lambda_i \geq 0 \right\},$$

a finitely generated polyhedral cone.

For $m = 2$, this structure matches the $1+1$-dimensional future Lorentz cone because

$$\mathcal{L}_2 = \text{cone}\{n_0, n_1\}, \qquad n_0 = (1, -1), \qquad n_1 = (1, 1).$$

The endpoint embedding is therefore

$$(u, v) \longmapsto \ell u \, n_0 + \ell v \, n_1 = \big(\ell(u+v), \ell(v-u)\big).$$

Each endpoint generator corresponds to an extreme null ray, and their positive combinations produce the complete future causal cone.

For spacetime dimension $D \geq 3$, however,

$$\mathcal{L}_D = \{(t, \mathbf{x}) : t \geq \|\mathbf{x}\|, \quad t \geq 0\}$$

has a continuously parameterized null boundary and is non-polyhedral [17]. The following theorem makes the resulting obstruction precise.

10.2 The Finite-Generator Obstruction

The situation changes in higher spacetime dimension.

Theorem 10.1 (Finite-generator obstruction)**.** *Let $D \geq 3$. No cone generated by finitely many relational extension vectors can equal the Lorentz cone*

$$\mathcal{L}_D = \{(t, \mathbf{x}) : t \geq \|\mathbf{x}\|, \ t \geq 0\}.$$

Proof. A cone generated by finitely many vectors,

$$C = \text{cone}\{g_1, \ldots, g_m\} = \left\{ \sum_{i=1}^m \lambda_i g_i : \lambda_i \geq 0 \right\},$$

is polyhedral and has at most finitely many extreme rays.

The extreme rays of $\mathcal{L}_D$ are its future-directed null rays. After normalizing the temporal component to unity, these rays are parameterized by

$$(1, \mathbf{n}), \qquad \mathbf{n} \in S^{D-2},$$

where

$$\|\mathbf{n}\| = 1.$$

For $D \geq 3$, the sphere S^{D-2} contains infinitely many elements. Hence $\mathcal{L}_D$ has infinitely many extreme rays and is not polyhedral. It therefore cannot be generated by a finite set of relational extension vectors. □

The theorem has an immediate consequence.

Corollary 10.1. *A finite alphabet with endpoint recombination by generator counting cannot recover the exact Lorentz cone of* $2+1$*- or* $3+1$*-dimensional Minkowski spacetime at a fixed discrete scale.*

Thus adding finitely many generator labels produces a polyhedral cone, not the continuously parameterized null boundary of a higher-dimensional Lorentz cone. A higher-dimensional extension must therefore alter the finite-generator mechanism rather than merely enlarge the alphabet.

10.3 Three Possible Routes to Higher-Dimensional Causal Structure

The obstruction theorem leaves three broad mathematical routes.

10.3.1 Continuously many relational directions

One may replace the finite generator alphabet by a direction space

$$\Sigma_D = S^{D-2}.$$

Each generator $\mathbf{n} \in S^{D-2}$ corresponds to a future-directed null extension

$$k(\mathbf{n}) = (1, \mathbf{n}).$$

A finite history is then

$$h = (\mathbf{n}_1, \ldots, \mathbf{n}_r),$$

with endpoint

$$Q(h) = \ell \sum_{j=1}^{r} k(\mathbf{n}_j).$$

Endpoint recombination identifies histories satisfying

$$h \sim_Q h' \quad \Longleftrightarrow \quad Q(h) = Q(h').$$

Because the positive hull of all future null directions is the future Lorentz cone,

$$\operatorname{cone}\left\{(1,\mathbf{n}) : \mathbf{n} \in S^{D-2}\right\} = \mathcal{L}_D,$$

this generator space reproduces the directional cone of D-dimensional Minkowski causality. At any fixed scale $\ell > 0$, however, endpoints of finite histories form the finite-sum set

$$\mathcal{E}_\ell^{(D)} = \left\{ \ell \sum_{j=1}^{r} k(\mathbf{n}_j) : r \in \mathbb{N},\ \mathbf{n}_j \in S^{D-2} \right\}. \tag{500}$$

Although the positive hull of these directions equals $\mathcal{L}_D$, the finite-history endpoints at fixed ℓ do not constitute the full continuum event structure. A refinement argument would still be required. Moreover, the assumed sphere S^{D-2}, its norm, and the map $\mathbf{n} \mapsto (1,\mathbf{n})$ already encode the target Lorentzian geometry. This route is therefore a representation, not yet a relational derivation.

10.3.2 Refining families of finite direction sets

A second route retains finite relational structures at every discrete scale. Let

$$V_r = \left\{\mathbf{n}_1^{(r)}, \ldots, \mathbf{n}_{m(r)}^{(r)}\right\} \subset S^{D-2}$$

be a finite set of directions, and define

$$C_r = \operatorname{cone}\left\{(1,\mathbf{n}) : \mathbf{n} \in V_r\right\}.$$

Each C_r is polyhedral, but if the direction sets become dense,

$$\overline{\bigcup_{r=1}^{\infty} V_r} = S^{D-2},$$

then the corresponding cones may converge to the Lorentz cone:

$$C_r \longrightarrow \mathcal{L}_D.$$

At finite resolution, Lorentz symmetry would be approximate. In the refinement limit, the set of admissible null directions could become rotationally complete.

The unresolved question is whether relational composition can generate the refinement $V_r \subset V_{r+1}$. If the direction sets are externally chosen to approximate S^{D-2}, the limiting symmetry is imposed; an intrinsic refinement rule could instead make it emergent.

10.3.3 An intrinsic non-polyhedral constraint

A third possibility is that higher-dimensional causal structure is not generated by a list of independent directions. Instead, admissibility may be determined by an emergent quadratic constraint.

Let V be an effective real vector space reconstructed from relational states. Suppose there exists a non-degenerate symmetric form

$$\eta : V \times V \to \mathbb{R}$$

with Lorentzian signature

$$(-, +, \ldots, +).$$

The future causal cone would then be characterized intrinsically by

$$\eta(v, v) \leq 0$$

together with a time orientation.

The required result would be a derivation of a Lorentzian quadratic invariant from relational composition. This would generate the continuous causal cone through one intrinsic constraint rather than an enumerated family of null directions, but no such derivation is presently available.

10.4 Dimensionality as an Interval-Growth Property

The binary witness has effective spacetime dimension two because causal interval cardinality grows quadratically with proper time:

$$|I(p, q)| \sim \frac{\tau^2(p, q)}{4\ell^2}.$$

For continuum Alexandrov intervals, volume scales with the Dth power of proper time, a relation that underlies several causal-set dimension estimators [11, 12, 13, 10]. Accordingly, a D-dimensional candidate continuum should satisfy asymptotically

$$N(\tau) \propto \tau^D,$$

where $N(\tau)$ is the number of relational events in a causal interval of effective duration τ.

This suggests defining an interval-growth dimension by

$$D_{\text{eff}}(\tau) = \frac{d \log N(\tau)}{d \log \tau},$$

or, for a discrete sequence of interval sizes,

$$D_{\text{eff}} \approx \frac{\log N(\tau_2) - \log N(\tau_1)}{\log \tau_2 - \log \tau_1}.$$

A viable higher-dimensional relational model must exhibit a scaling regime in which

$$D_{\text{eff}}(\tau) \longrightarrow D$$

for finite D. Branching alone is insufficient. Exponential growth,

$$N(\tau) \sim e^{c\tau},$$

would characterize a tree-like hierarchy rather than a finite-dimensional Lorentzian spacetime. For a model intended to approximate a regular finite-dimensional Lorentzian continuum over an appropriate scaling regime, polynomial interval growth is therefore a necessary dimensional diagnostic.

For the observed macroscopic case, the target is

$$N(\tau) \propto \tau^4, \qquad D_{\text{eff}} \longrightarrow 4.$$

Obtaining this scaling would support an effective $3+1$-dimensional interpretation, but it would not by itself establish a Lorentzian metric. The model must also reproduce null boundaries, causal ordering, and the appropriate interval-shape relations.

10.5 Curvature as Relationally Varying Causal Organization

In the flat witness, every endpoint has the same two future covers:

$$(u, v) \lessdot_R (u+1, v), \qquad (u, v) \lessdot_R (u, v+1).$$

Conditional on the Local-Propagation Principle, these covers are interpreted as elementary future propagation steps. The resulting cover structure is translation invariant, and interval cardinality depends only on endpoint differences.

Curvature would not require the local future cones at different events to have different Lorentzian types. In continuum Lorentzian geometry, local tangent-space cones may remain mutually isomorphic even when the spacetime is curved. What fails is the existence of one globally uniform identification of neighboring causal structures.

Let

$$\mathcal{C}^+(p)$$

denote the admissible future cone at a relational event p. Flatness may be represented schematically by the existence of a common translation rule identifying

$$\mathcal{C}^+(p) \cong \mathcal{C}^+(q)$$

uniformly for all suitable p and q, together with translation-invariant interval growth.

An emergent curved geometry could instead retain the same local cone type,

$$\mathcal{C}^+(p) \cong_{\text{loc}} \mathcal{C}^+(q),$$

while allowing the comparison of neighboring cones to acquire nontrivial event dependence. Schematically, comparison maps

$$\mathsf{T}_{p\to q} : \mathcal{C}^+(p) \longrightarrow \mathcal{C}^+(q)$$

need not arise from one global translation rule. Their composition around relational loops could exhibit nontrivial transport, and causal interval growth could depend on location and direction as well as endpoint separation.

The relevant distinction is therefore

local Lorentzian cone type $\neq$ globally uniform cone identification, curvature $\sim$ nontrivial transport and interval-volume dependence.

In continuum Lorentzian geometry, the volume of a sufficiently small causal diamond differs from its flat-spacetime value by curvature-dependent corrections [10]. Related causal-set analyses extract curvature information from finite order intervals and chain abundances in causal diamonds [18, 19].

After an effective metric and proper-time function have independently been supplied, one may define the diagnostic

$$\Delta N_{\mathrm{diag}}(p,q) = N(p,q) - N_{\mathrm{flat}}\big(\tau(p,q)\big).$$

A systematic nonzero value could then be compared with effective curvature data:

$$\Delta N_{\mathrm{diag}}(p,q) \longleftrightarrow \text{effective curvature correction.}$$

This is a diagnostic after metric specialization, not a derivation of curvature from relational counting alone, because $\tau(p,q)$ already presupposes metric information.

A genuinely relational reconstruction would first require an independent order-theoretic or counting-based estimator

$$\widehat{\tau}_{\mathrm{rel}}(p,q)$$

that does not presuppose the metric being reconstructed. Only then could a residual such as

$$\Delta N_{\mathrm{rel}}(p,q) = N(p,q) - N_{\mathrm{flat}}\big(\widehat{\tau}_{\mathrm{rel}}(p,q)\big)$$

serve as candidate input to a relational curvature estimator. No such independent estimator or curvature operator has yet been derived from the present history and endpoint rules.

Even a successful curvature estimator would establish only a kinematical representation of curved geometry. A physical theory would additionally require dynamical laws governing how relational content changes causal transport, interval volume, and effective curvature.

10.6 From Admissible Structure to Dynamics

The present construction determines which histories and endpoints are structurally available, but it assigns no amplitudes, probabilities, or dynamical weights to them. Let

$$\mathfrak{R}$$

denote the class of admissible relational realizations. A dynamics would require an additional object such as

$$W : \mathfrak{R} \to \mathbb{R}_{\geq 0},$$

or, in a quantum formulation,

$$\mathcal{A} : \mathfrak{R} \to \mathbb{C},$$

subject to appropriate normalization, composition, and invariance conditions.

The distinction is:

$$\text{admissibility} = \text{which structures are permitted},$$

whereas

$$\text{dynamics} = \text{how permitted structures are weighted or transformed.}$$

PGND and the current bridge principles constrain the first question. They do not yet determine the second.

A candidate dynamics would need to explain at least:

1. why persistent relational extensions occur;
2. why particular recombination classes become physically relevant;
3. why approximately local causal structures dominate;
4. why a finite macroscopic dimension is stable;
5. why Lorentz-violating microscopic features become suppressed; and
6. how matter-like and geometry-like relational patterns interact.

Until such a rule is supplied, the physical world can be described as one admissible expression of relational distinction, but its actual realization has not been derived.

10.7 Result of the Generalization Audit

The finite-generator theorem establishes that the binary mechanism is exact in $1+1$ dimensions because the corresponding future Lorentz cone has two extreme null rays. In $D \geq 3$, exact recovery requires continuously many directions, a refining family of finite direction sets, or an intrinsic non-polyhedral constraint. Each route introduces structure not supplied by the present binary construction.

The remaining mathematical question is therefore:

> *Can a PGND-compatible relational extension rule generate a finite-dimensional non-polyhedral causal cone with Lorentzian interval scaling, without presupposing a background sphere, metric, or spacetime dimension?*

A positive answer would provide a route toward higher-dimensional emergence; a negative answer would identify additional geometric structure as indispensable.

11 Conclusion

This paper has constructed one explicit witness showing that primitive global non-distinction is compatible with relationally differentiated Lorentzian causal organization. PGND serves as a restrictive primitive-description constraint rather than a generative axiom: the history space, endpoint quotient, causal order, and metric realization arise only under the independently stated bridge principles and physical specializations identified by the dependency and countermodel audits.

Under those assumptions, same-occupant articulation develops into the binary history monoid $\mathcal{H} = \{0,1\}^*$, whose count-based endpoint quotient is $\mathcal{E} \cong \mathbb{N}^2$. Extension-induced reachability gives the locally finite product order, and, for every fixed $\ell > 0$,

$$\iota_\ell(u,v) = \big(\ell(u+v), \ell(v-u)\big)$$

identifies that order with the causal order induced on a regular discrete $1+1$-dimensional Minkowski future wedge. The mathematical contribution is the audited relational route to this familiar null-lattice structure, not the independent discovery of the product order or its null-coordinate realization.

The separate refinement theorem establishes that, for causally ordered lattice approximants converging to fixed strictly timelike continuum endpoints,

$$\lim_{\ell\to 0} 2\ell^2 \big|I[p_\ell, q_\ell]\big| = V_{\mathrm{Alex}}(P,Q) = \frac{\tau(P,Q)^2}{2}.$$

Independently, the large-interval ordering fraction approaches the 1+1-dimensional Myrheim–Meyer value. This is a controlled interval-volume limit, not a proof of convergence for arbitrary observables or an exact microscopic Lorentz symmetry.

The higher-dimensional audit also fixes the boundary of the construction. The binary mechanism is exact in $1+1$ dimensions because the future Lorentz cone has two extreme null rays. In spacetime dimension $D \geq 3$, the Lorentz cone is non-polyhedral and cannot be generated by a fixed finite collection of endpoint directions. An exact extension must therefore introduce continuously many directions, a refining family of finite direction sets, or an intrinsic non-polyhedral constraint.

The result establishes admissibility, not physical realization, uniqueness, or empirical adequacy. No dynamics presently selects the bridge principles or produces higher-dimensional geometry, curvature, matter, quantum structure, or testable consequences. The remaining foundational question is whether those principles can be derived or unified by an intrinsic relational dynamics that also supplies the missing physical structures. The present construction does not answer that question, but makes its assumptions and mathematical obstructions explicit.

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Use of AI-Assisted Tools AI-assisted language tools were used for language editing and formatting support. The author takes full responsibility for all scientific content and final manuscript decisions.

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